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التاريـــخ: ٢٠١١/١٢/٤

نموذج رقم (١٨)
اقرار والتزام بالمعايير الأخلاقية والأمانة العلمية وقوانين الجامعة الأردنية وأنظمتها وتعليماتها لطلبة الماجستير

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عنوان الرسالة:
The Application of Spline Functions to Fractional Differential Equations

اعلن بانني قد التزمت بقوانين الجامعة الأردنية وأنظمتها وتعليماتها وقراراتها السارية المفعول المتعلقة باعداد رسائل الماجستير عندما قمت شخصيا" باعداد رسالتي وذلك بما ينسجم مع الأمانة العلمية وكافة المعايير الأخلاقية المتعارف عليها في كتابة الرسائل العلمية. كما انني اعلن بأن رسالتي هذه غير منقولة او مستلة من رسائل او كتب او ابحاث او اي منشورات علمية تم نشرها او تخزينها في اي وسيلة اعلامية، وتأسيسا على ما تقدم فانني اتحمل المسؤولية بأنواعها كافة فيما لو تبين غير ذلك بما فيه حق مجلس العمداء في الجامعة الأردنية بالغاء قرار منحي الدرجة العلمية التي حصلت عليها وسحب شهادة التخرج مني بعد صدورها دون أن يكون لي اي حق في التظلم أو الاعتراض أو الطعن بأي صورة كانت في القرار الصادر عن مجلس العمداء بهذا الصدد.

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THE APPLICATION OF SPLINE FUNCTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

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This Thesis was submitted in Partial Fulfillment of the requirements for the Degree of Master's Degree of Mathematics

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COMMITTEE DECISION

This Thesis (The application of Spline Functions to Fractional Differential Equations) was successfully defended and approved on December, 2011.

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DEDICATION

TO MY FAMILY, HUSBAND KHALED AND FRIENDS

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THE APPLICATION OF SPLINE FUNCTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this thesis, we use the spline functions of polynomial form to approximate the solution of linear and nonlinear fractional differential equations. The description of the proposed spline approximation method is introduced. The error analysis and stability of the method are theoretically investigated. Numerical examples are given to illustrate the applicability, accuracy and stability of the proposed scheme. The fractional derivatives are defined in the Caputo sense. The results obtained are in good agreement with the exact analytical solutions and the numerical results presented elsewhere.

Results obtained using the scheme presented here show that the spline function method is very powerful and efficient technique in finding numerical solutions of fractional differential equations and it increases the accuracy of the solutions. The algorithm described in this paper is expected to be further employed to solve similar nonlinear problems in fractional calculus.

Chapter One

Introduction

1.1 Statement of the Problem

Mathematical modeling of complex processes is a major challenge for contemporary scientists. In contrast to simple classical systems, where the theory of integer-order differential equations is sufficient to describe their dynamics, complex systems are characterized by the variability of structures in them, mutiscale behavior and nonlinearity in the mathematical description of the mutual relationship between parameters (Leszczynski, 2011). Recent investigations have shown that many complex physical and biological systems can be represented more accurately through fractional derivative formulation (Mainardi, 1997) Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various complex materials and systems. Therefore, the number of scientific and engineering problems involving fractional derivatives is already very large and still growing and perhaps the fractional calculus (that is, derivatives and integrals of any real or complex order) will be the calculus of the twenty-first century ((Leszczynski, 2011), (Mainardi, 1997), (Caputo, 1967) and (Kilbas, et al., 2006).

For example, fractional calculus has been successfully applied to problems in system biology, physics, chemistry and biochemistry, hydrology, medicine, and finance. In many cases these new fractional-order models are more adequate than the previously used integer-order models, because fractional derivatives and integrals enable the description of the memory and hereditary properties inherent in various materials and processes that are governed by anomalous diffusion. Hence, there is a growing need to find the solution behavior of these fractional differential equations. However, the analytic solutions of most fractional differential equations generally cannot be obtained. As a consequence, approximate and numerical techniques are playing an

important role in identifying the solution behavior of such fractional equations and exploring their applications. The main objective of this thesis is to extend the application of the spline functions of polynomial form (Diethelm & Freed, 1999), (Diethelm & Walz, 1997) and (Ramadan, 2005) to solve linear and nonlinear differential equations of fractional order. The spline functions of polynomial form was successfully used to solve system analysis (Micula, 1987) delay differential equations (Ramadan, et al., 2005) and (Ramadan, 2000) and delay differential equations of fractional order (Ramadan, et al., 2011)

The analysis of fractional differential equation (existence, uniqueness, and stability) of the form:

$$y^{(\propto)}(x) = f\big(x, y(x)\big), \qquad y(0) = y_0 \tag{1.1}$$

is studied by (Diethelm & Ford, 2003). A number of approximate solutions of the initial value problem (1.1) have been proposed in the literature, where the Adams-Bashforth-Moulton method is introduced in (Diethelm, 1999). An alternative is the backward differentiation formula, presented in (Diethelm, 1997). where the idea of this method is based on discretizing the differential operator in the fractional differential equation (1.1) by certain finite difference. The main result of (Diethelm, 1979). was that under suitable assumption we can expect an $o(h^{\alpha-2})$ convergence behavior. In (Ramadan, 2005) an improvement of the performance of the method presented in (Diethelm, 1997). is achieved by applying Extrapolation principles. Kia Dithelm et. al. in (Diethelm, et al., 2006). considered a fast algorithm for the numerical solution of initial value problems of the form (1.1) in the sense of Caputo and discuss potential problems in the development of generally applicable schemes. More recently, Lagrange multiplier method and the homotopy perturbation method are used to solve numerically multi-order fractional differential equation, see (Sweilam, et al., 2007)

Ramadan, introduced in (Ramadan, 2005) the solution of the first order delay differential equation of the form:

$$y'(x) = f\left(x, y(x), y\left(g(x)\right)\right) \qquad a \leq x \leq b$$

$$y(a) = y_0 \,, y(x) = \varphi(x) \;\;, x \in [a^*, \mathrm{a}],$$

Using the spline function of the polynomial form.

Our purpose is to extend and generalize the polynomial Spline functions used in (Ramadan, 2005) and (Ramadan, et al., 2011) to approximate the solution of the fractional ordinary differential equation. The fractional derivative was defined in the Caputo sense. This method is very powerful and efficient technique in finding numerical solutions for wide classes of problems and it increases the accuracy of the solutions.

1.2 Motivation

The motivation of this work is to present a new method for solving fractional differential equations. The new method is based on spline function of polynomial form.

1.3 Synopsis of the Thesis

Following this introductory chapter, a review of the fractional calculus theory and spline function are presented in chapter two. In chapter three, we implement the spline function method to solve fractional differential equations. We give two nonlinear problems in order to illustrate the applicability, accuracy and stability of the proposed method. Finally, the conclusion of this study is presented

Chapter Two

Fundamentals of Fractional Calculus Theory and Spline Function

2.1 Fundamentals of Fractional Calculus Theory

2.1.1 Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. The term "Fractional" is misnomer, but it is retained following the prevailing use. Fractional calculus is also considered as a level of calculus that generalizes the notion of derivative of function of integer to one with non-integer order, allowing calculations, such as deriving a function to ½ orders. Despite "generalized" would be a better option; the name ‘‘fractional’’ is used for denoting this kind of derivatives.

The origins can be traced back to the end of the 17^{th} century, the time when Newton and Leibniz developed the foundations of the differential and integral calculus. In particular, Leibniz introduced the symbol $\frac{d^n}{dx^n}f(x)$ to denote the n^{th} derivative of a function f.When he reported this in a letter to de L'Hospital (apparently with the implicit assumption that), de L'Hospital replied: ‘what does $\frac{d^n}{dx^n}f(x)$ mean if n=1/2?". This letter from de L'Hospital, written in 1695, is nowadays commonly accepted as the first occurrence of what we today call a fractional derivative (Diethelm, 2009).

However, a rigorous investigation was first carried out by Liouville ,where he defined the first outcast of an operator of fractional integration. Later investigation and further envelopments by others, Riemann led to the construction of the integral-based Riemann-Liouville fractional operator which has seen a valuable cornerstone in fractional calculus ever since.

Prior to Liouville and Riemann, Euler took the first step in the study of fractional integration when he studied the simple case of fractional integrals of monomial of arbitrary real order in the heuristic fashion of the time. It has been said to have led him to construct Gamma function for fractional powers of the factorial.

An early attempt by Liouville was later purified by Swedish mathematician Holmgren, who in 1865 made important contributions in the growing study of fractional calculus. But it was Riemann who reconstructed it to fit Abel's integral equation and thus made it vastly more useful (Munkhammer, 2004).

However, it may be considered a novel topic as well, since only from a little more than three decades it has been object of specialized conferences and treatises. For the first conference, the merit is ascribed to Ross, B. who organized the First Conference on Fractional Calculus and its Applications at the University of New Haven in June 1974, and edited the proceedings. For the first monograph, the merit is ascribed to K.B.Oldham and J.Spanier, who after a joint collaboration started in 1968, published a book devoted solely or partly to fractional calculus and its applications includes about a dozen of titles, among which the encyclopedic treatise by Samko, Kilbas and Marichev is the most prominent (Gorenflo and Mainardi, 1997).

Today there exist many different forms of fractional integral operators, ranging from divided difference types to infinite-sum types, such as Riemann-Liouville, Caputo, Grünwald-Letnikov,…etc; but the Riemann-Liouville operator is still the most frequently used when fractional integration (or derivative) is performed.

In recent years considerable interest in fractional calculus has been stimulated by the applications that this calculus finds in numerical analysis and different areas of physics (e.g., it can be used to model anomalous diffusion in Hamiltonian Chaos fractional partial

differential equation. Other applications in physics involve fractional mechanics and fractional oscillators).

Applications of fractional calculus in general also appear in speculative option valuation in finance and are related to so called heavy tails in electrical engineering (Munkhammar, 2004).

In the last few years, fractional calculus, also, was applied successfully in various areas, e.g. chemistry, biology, modeling and identification, electronics and others.

Unlike the integral order derivative, the fractional order derivative at a point x is not determined by an arbitrary small neighborhood of x. This follows from the classical representation of D^n as a limit of a difference quotient. However, to calculate $D_a^{\propto} f(x)$, for $\propto \notin N$, the definition tells us that we need to know f throughout the entire interval $[a, x]$.In other words, the fractional derivative is not local property of the function. This obscures the geometric meaning of these fractional derivatives. The non-local property is the most

important advantage of using fractional differential equations. Because of this property, fractional derivatives provide a powerful tool for the description of memory and hereditary properties of different substances.

The basic idea behind fractional calculus is intimately related to the fundamental theorem of calculus.

Theorem 2.1.1[The Fundamental Theorem of Calculus]

Let $f: [a, b] \to R$ be a continuous function, and let $F: [a, b] \to R$ be defined by $F(x) := \int_a^x f(t)dt$

Then, F is differentiable and $F' = f$

Therefore, we have a close relation between differential operators and integral operators. It is one of the goals of fractional calculus to retain this relation in a suitably generalized sense.

Definition 2.1.1:

1. By *D*, we denote the operator that maps a differentiable function onto its derivative, i.e.$Df(x) := f^{'}(x)$

2. By J_a , we denote the operator that maps a function, assumed to be (Riemann) integrable on the compact interval [a, b], onto its primitive centered at, i.e.

$$J_a f(x) := \int_a^x f(t)dt \quad ,for\ a \leq x \leq b \qquad (2.1.1)$$

3. For $n \in N$, we use the symbols $D^n\ and\ J_a^n$ to denote the n-fold iterates of $D\ and\ J_a$, respectively,i.e. we set $D^1 := D, J_a^1 = J_{a,} and\ D^n := DD^{n-1}\ and\ J_a^n := J_a J_a^{n-1}\ for\ n \geq 2$

Note that from Theorem 2.1.1 we have

$$DJ_a f = f \qquad (2.1.2)$$

Which implies that

$$D^n J_a^n f = f\ \ for\ n \in N \qquad (2.1.3)$$

i.e. D^n is the left inverse of J_a^n in a suitable space of functions.

2.1.2 The Higher Transcendental Functions Gamma and Beta

Definition 2.1.2

The Gamma function is a generalization of the factorial function and is given by

$$\Gamma(x) := \int_0^{\infty} e^{-t} t^{x-1}\, dt \quad for\ x > 0 \qquad (2.1.4)$$

The Gamma function satisfies the following two properties:

1 $\Gamma(x) = (x-1)!, \forall x \in Z^+$

2 $\Gamma(x+1) = x\Gamma(x), \forall x$

Definition 2.1.3

The Beta function is a generalization of the reciprocal of a binomial coefficient and is given by

$$B(x,y) := \int_0^1 \tau^{x-1}(1-\tau)^{y-1}\, d\tau \qquad (2.1.5)$$

The Beta function can be written in terms of the gamma function, as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \qquad (2.1.6)$$

Remark 2.1.1: $\Gamma(1/2) = \sqrt{\pi}$

Proof: Use the substitution $t=x^2$

$$\Gamma(1/2) = \int_0^{\infty} e^{-t} \frac{dt}{\sqrt{t}} = 2\int_0^{\infty} e^{-x^2}\, dx$$

$$(\Gamma\left(\frac{1}{2}\right))^2 = 4 \int_0^{\infty}\int_0^{\infty} e^{-x^2-y^2}\, dxdy$$

Using the polar coordinates

$x = r\cos\varphi, y = r\sin\varphi$

$$(\Gamma\left(\frac{1}{2}\right))^2 = 4\iint_{0\,0}^{\infty\,\pi/2} e^{-r^2} r\,d\varphi dr = 2\pi\int_0^{\infty} e^{-r^2} r dr = \pi\int_0^{\infty} e^{-u}\,du = \pi$$

So $\Gamma(1/2) = \sqrt{\pi}$

2.1.3 Left Riemann-Liouville Fractional Operators

The left n=fold integral operator on the interval (a,b) is defined as the solution to the initial value problem (Miller and Ross, 1993)

$D^n u(x) = f(x), \forall x \in (a,b)$

$u^{(i)}(a) = 0\,, i = 0, \ldots\ldots, n-1$ (2.1.7)

By n-fold here, means that the integration is deployed n-times. To see how a general form of the solution of (2.1.7) unfolds, first we write the solution of (2.1.7) for n=1 as

$u(x) = \int_a^x f(t)dt$ (2.1.8)

Where we have integrated both sides from $a\ to\ x$ and applied the initial condition $u(a) = 0$ .Next, we proceed in a similar manner for the case n=2. We integrate both sides of $D^2 u(x) = f(x)$ twice to obtain

$\iint_{a\,a}^{x\,t} f(\xi)d\xi dt = \iint_{a\,a}^{x\,t} D^2\,u(\xi)d\xi dt = u(x) - (x-a)u^{'}(a) - u(a) = u(x)$ (2.1.9)

To reduce the left hand side of (2.1.9) to a single integration, we interchange the order of integration

$$u(x) = \iint_{a\,\xi}^{x\,x} f(\xi)dtd\xi = \int_a^x (x-\xi)f(\xi)d\xi \qquad (2.1.10)$$

Repeating this process for general, we find that the solution of (2.1.7) is given by the formula

$$u(x)=\int_a^x\int_a^{\xi 1} \ldots\ldots \int_a^{\xi_{n-1}} f(\xi)d\xi d\xi_{n-1}\ldots\ldots d\xi_1=\int_a^x \frac{(x-\xi)^{n-1}}{(n-1)!}f(\xi)d\xi \qquad (2.1.11)$$

Example 2.1.1 Let $f(x)=x, n=3, a=0\ (to\ remove\ residue\ term)$ the last equation becomes

$$\int_0^x\int_0^{\xi 1}\int_0^{\xi 2} \xi d\xi d\xi_2\, d\xi_1=\frac{1}{2!}\int_0^x \frac{\xi}{(x-\xi)^{-2}}d\xi=\frac{x^4}{4!}\ (the\ 3-fold\ integral\ of\ u(x)=x)$$

Since $(n-1)!=\boldsymbol{\Gamma(n)}$, Riemann realized that the right hand side of equation (2.1.11) has meaning even when n takes non-integer values. Thus, perhaps it was natural to define the left Riemann-Liouville fractional integral as follows.

Definition 2.1.4 (Miller and Ross, 1993) [Left Riemann-Liouville Fractional integral]

Let $f$be defined on the interval $\propto\in R^+$.Then the left Riemann-Liouville fractional Integral of order $\propto$ is defined to be

$$J_{a^+}^{\propto}f(x):=\frac{1}{\Gamma(\propto)}\int_a^x (x-t)^{\propto-1}f(t)dt \qquad (2.1.12)$$

Note that for $\propto=0, we\ set\ J_{a^+}^{\propto}:=I\ (the\ identity)$

Example 2.1.2 Let $f(x)=(x-a)^{\beta}\ , for\ some\ \beta>-1\ and\ \propto>0. Find\ J_{a^+}^{\propto}f(x).$

Solution:

$$J_{a^+}^{\propto} f(x) := \frac{1}{\Gamma(\propto)} \int_a^x (x-t)^{\propto-1} (t-a)^{\beta} dt$$

Now, use the substitution $t = a + s(x-a) \gg x - t = (x-a)(1-s)$

$$J_{a^+}^{\propto} f(x) := \frac{1}{\Gamma(\propto)} \int_0^1 (x-a)^{\propto-1} (x-a)^{\beta} (1-s)^{\propto-1} (x-a) s^{\beta}\, ds$$

Where if $t = a \gg s = 0 \; and \; if \; t = x \gg s = 1$

$$J_{a^+}^{\propto} f(x) := \frac{1}{\Gamma(\propto)} \int_0^1 (x-a)^{\propto+\beta} (1-s)^{\propto-1} s^{\beta}\, ds$$

$$\frac{(x-a)^{\propto+\beta}}{\Gamma(\propto)} \int_0^1 (1-s)^{\propto-1} s^{\beta}\, ds$$

Using Eq (2.1.6), $B(\beta+1, \propto) := \int_0^1 (1-s)^{\propto-1} s^{\beta}\, ds \;=\; \frac{\Gamma(\beta+1)\Gamma(\propto)}{\Gamma(\beta+\alpha+1)}$

So $\quad J_{a^+}^{\propto} f(x) := \frac{1}{\Gamma(\propto)} \frac{\Gamma(\beta+1)\Gamma(\propto)}{\Gamma(\beta+\alpha+1)} \; (x-a)^{\beta+\alpha} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} \; (x-a)^{\beta+\alpha}$

A table of some fractional integrals is given below.

$f(x)$	$J_{a^+}^{\propto} f(x)$	specification
$(x-a)^{\beta}$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} \; (x-a)^{\beta+\alpha}$	$a \in C, \propto > 0, Re(\beta) > -1$
c	$\frac{c}{\Gamma(\alpha+1)} \; (x-a)^{\alpha}$	$a \in R, \propto > 0$
e^{bx}	$b^{-\alpha} e^{bx}$	$a = -\infty, \propto > 0, Re(b) > 0$

Table 1: Left Riemann-Liouville fractional integrals of some functions

Theorem 2.1.2 (Munkhammer, 2004) for any $f \in C[a,b]$ the left Riemann-Liouville fractional operators satisfy

$$J_{a^+}^{\propto} J_{a^+}^{\beta} f(x) = J_{a^+}^{\propto+\beta} f(x), \forall \propto, \beta > 0 \qquad (2.1.13)$$

Proof: Using (2.1.12), and interchanging the order of integration, we have

$$J_{a^+}^{\propto} J_{a^+}^{\beta} f(x) = \frac{1}{\Gamma(\propto)\Gamma(\beta)} \int_a^x (x-t)^{\propto-1} \int_a^t (t-s)^{\beta-1} f(s) \quad ds\, dt$$

$$= \frac{1}{\Gamma(\propto)\Gamma(\beta)} \int_a^x f(s) \int_s^x (x-t)^{\propto-1} (t-s)^{\beta-1}\, dt\, ds$$

To evaluate the inner integral, we make the change of variables $\tau = \frac{t-s}{x-s}. Therefore,$

$$J_{a^+}^{\propto} J_{a^+}^{\beta} f(x) = \frac{1}{\Gamma(\propto)\Gamma(\beta)} \int_a^x (x-s)^{\propto+\beta-1} f(s) \int_0^1 (1-\tau)^{\alpha-1} \tau^{\beta-1}\, ds\, dt$$

$$= \frac{B(\propto,\beta)}{\Gamma(\propto)\Gamma(\beta)} \int_a^x (x-s)^{\propto+\beta-1} f(s) ds = \quad J_{a^+}^{\beta+\propto} f(x)$$

Theorem 2.1.2 is known as the semi-group property for the left fractional integral operators.

There exist a number of definitions of the fractional order derivative. The classical one is the definition by Riemann-Liouville fractional derivatives have a singularity at zero. This is why differential equations involving these derivatives require initial conditions of special form lacking clear physical interpretation. These shortcomings do not occur with the regularized fractional derivative in the sense of Caputo as an example.

Definition 2.1.5 (Miller and Ross, 1993) [Left Riemann-Liouville Fractional Derivative] Let *f* be defined on the interval (a, b) and $\propto \in R^{+}$ and $n = \lceil \propto \rceil$, $(n-1 \leq \propto \leq n)$. Then the Left Riemann- Liouville fractional derivative of order $\propto$ is defined to be

$$D_{a^+}^{\propto} f(x) := D^n J_{a^+}^{n-\propto} f = \frac{d^n}{dx^n} \left(\frac{1}{\Gamma(n-\propto)} \int_a^x (x-t)^{n-\propto-1} \ f(t)dt \right). \qquad (2.1.14)$$

The newly defined operator $D_{a^+}^{\propto}$ coincides with the classical differential operator D^n whenever$\propto \in N$.

Example 2.1.3 Let $f(x) = (x-a)^{\nu} \ for \ some \ \nu > -1 \ and \ \propto > 0. Find \ D_{a^+}^{\propto} f(x)$

Solution: The left fractional derivative of $(x-a)^{\nu}$ is given by

$$D_{a^+}^{\propto} f(x) = \frac{d^n}{dx^n} \ J_{a^+}^{n-\propto} (x-a)^{\nu} = \frac{d^n}{dx^n} \frac{\Gamma(\nu+1)(x-a)^{n-\propto+\nu}}{\Gamma(\nu+n-\propto+1)}$$
$$= \frac{\Gamma(\nu+1)(x-a)^{-\propto+\nu}}{\Gamma(\nu-\propto+1)}$$

Example 2.1.4 Let $f(x) = c, \propto = \frac{1}{2}, a = 0.$

$$D_{0^+}^{1/2} c = D \ J_{0^+}^{1/2} c = \frac{d}{dx}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x (x-t)^{-\frac{1}{2}} c dt \right] = \frac{c}{\sqrt{\pi}} \frac{d}{dx} \left[\int_0^x u^{-\frac{1}{2}} \ du \right] = \ \frac{c}{\sqrt{\pi}} x^{-1/2}$$

2.1.4 Right Riemann-Liouville Fractional Operators

The right n-fold integral operator on the interval *(a, b)* is defined as the solution to the initial value problem (Miller and Ross, 1993)

$(-D)^n u(x) = f(x), \forall x \in (a,b)$

$$u^{(i)}(b) = 0\ , i = 0, \ldots\ldots, n-1 \qquad (2.1.15)$$

To see how a general form of the solution of (2.1.15) unfolds, first we write the solution of (2.1.15) for n=1 as

$$u(x) = \int_x^b f(t)dt \qquad (2.1.16)$$

Where we have integrated both sides from x to b and applied the initial condition u (b) =0. Next, we proceed in a similar manner for the case n=2.

We integrate both sides of $D^2u(x) = f(x)$ twice to obtain

$$\int_x^b\int_t^b f(\xi)d\xi dt = \int_x^b\int_t^b D^2u(\xi)d\xi dt = u(x) - (x-a)u'(b) - u(b) = u(x) \qquad (2.1.17)$$

To reduce the left hand side of (2.1.17) to a single integration, we interchange the order of integration

$$u(x) = \int_x^b\int_x^\xi f(\xi)dtd\xi = \int_x^b (\xi - x)f(\xi)d\xi \qquad (2.1.18)$$

Repeating this process for general $n \in N$,we find that the solution of (2.1.15) is given by the formula

$$u(x) = \int_x^b\int_{\xi 1}^b \ldots\ldots\ldots \int_{\xi_{n-1}}^b f(\xi)d\xi d\xi_{n-1} \ldots\ldots. d\xi_1 = \int_x^b \frac{(\xi - x)^{n-1}}{(n-1)!} f(\xi)d\xi \qquad (2.1.19)$$

Definition 2.1.4 (Miller and Ross, 1993) [Right Riemann-Liouville Fractional integral]

Let f be defined on the interval (a , b) $and \ \propto \in R^{+}$.Then the right Riemann-Liouville fractional Integral of order $\propto$ is defined to be

$$J_{b-}^{\propto} f(x) := \frac{1}{\Gamma(\propto)} \int_{x}^{b} (t-x)^{\propto-1} f(t)\, dt \qquad (2.1.20)$$

Definition 2.1.7 (Miller and Ross, 1993) [Right Riemann-Liouville Fractional Derivative] Let f be defined on the interval *(a, b)* and $\propto \in R^{+}$ and $n = \lceil \propto \rceil$, $(n-1 \leq \propto \leq n)$. Then the right Riemann- Liouville fractional derivative of order $\propto$ is defined to be

$$D_{b-}^{\propto} f(x) := (-D)^{n} J_{b-}^{n-\propto} f(x) = (-1)^{n} \frac{d^{n}}{dx^{n}} \left(\frac{1}{\Gamma(n-\propto)} \int_{x}^{b} (t-x)^{n-\propto-1} \ f(t)\, dt \right). \qquad (2.1.21)$$

Example 2.1.5 [Right Integral and Derivative of $(b-x)^{\beta}$] A closed form expression may be obtained for the right fractional integral of $f(x) = (b-x)^{\beta}$.Applying the definition (2.1.6), we have

$$J_{b-}^{\propto} f(x) := \frac{1}{\Gamma(\propto)} \int_{x}^{b} (t-x)^{\propto-1} (b-t)^{\beta}\, dt$$

Using the substitution $\tau = \frac{t-x}{b-x}, we\ have$

$$J_{b-}^{\propto} (b-x)^{\beta} := \frac{(b-x)^{\beta+\propto}}{\Gamma(\propto)} \int_{0}^{1} \tau^{\propto-1} (1-\tau)^{\beta}\, d\tau$$

$$= \frac{(b-x)^{\beta+\propto}}{\Gamma(\propto)} B\ (\propto, \beta+1) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}\ (b-x)^{\beta+\alpha}$$

The fractional derivative of $(b-x)^{\beta}$ is given by

$$D_{b^-}^{\propto}(b-x)^{\beta} = (-1)^n \frac{d^n}{dx^n} J_{b^-}^{n-\propto}(b-x)^{\beta} = (-1)^n \frac{d^n}{dx^n} \frac{1}{\Gamma(\propto)} \int_x^b (t-x)^{n-\propto-1}(b-t)^{\beta} dt$$

$$= (-1)^n \frac{d^n}{dx^n} \frac{\Gamma(\beta+1)(b-x)^{\beta+n-\propto}}{\Gamma(\beta+n-\propto+1)} = \frac{\Gamma(\beta+1)(b-x)^{\beta-\propto}}{\Gamma(\beta-\propto+1)}$$

A table of some fractional derivatives is given below

$f(x)$	$D_{a^+}^{\propto} f(x)$	Specification
$(x-a)^{\beta}$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha}$	$a \in C, \propto > 0, Re(\beta) > -1$
c	$\frac{c}{\Gamma(1-\alpha)} (x-a)^{-\alpha}$	$c \in C, \propto > 0$
e^{bx}	$b^{\alpha} e^{bx}$	$a = -\infty, \propto > 0, Re(b) > 0$

Table 2: Left Riemann-Liouville fractional derivatives of some functions

Theorem 2.1.2 (Munkhammer, 2004) for any $f \in C[a,b]$ the right Riemann-Liouville fractional operators satisfy

$$J_{b^-}^{\propto} J_{b^-}^{\beta} f(x) = J_{b^-}^{\propto+\beta} f(x), \forall \propto, \beta > 0 \qquad (2.1.22)$$

Proof: Using (2.1.20), and interchanging the order of integration, we have

$$J_{b^-}^{\propto} J_{b^-}^{\beta} f(x) = \frac{1}{\Gamma(\propto)\Gamma(\beta)} \int_x^b (x-t)^{\propto-1} \int_t^b (s-t)^{\beta-1} f(s) \quad ds\, dt$$

$$= \frac{1}{\Gamma(\propto)\Gamma(\beta)} \int_x^b f(s) \int_x^s (t-x)^{\propto-1} (s-t)^{\beta-1}\, dt\, ds$$

To evaluate the inner integral, we make the change of variables $\tau = \frac{t-x}{s-x}. Therefore,$

$$J_{b^-}^{\propto} J_{b^-}^{\beta} f(x) = \frac{1}{\Gamma(\propto)\Gamma(\beta)} \int_x^b (s-x)^{\propto+\beta-1} f(s) \int_0^1 \tau^{\alpha-1} (1-\tau)^{\beta-1} \, d\tau \, ds$$

$$= \frac{B(\propto,\beta)}{\Gamma(\propto)\Gamma(\beta)} \int_x^b (s-x)^{\propto+\beta-1} f(s) ds = \quad J_{b^-}^{\beta+\propto} f(x)$$

Theorem 2.1.3 is known as the semi-group property for right fractional integral operators (Miller and Ross, 1993).

The right and left fractional derivatives at a point are generally not equal, and the value of the left Riemann-Liouville fractional derivatives at a point x depends on the function values at all points in the interval [a,x], and that for the right Riemann-Liouville fractional derivatives at a point x depends on the function values at all points in the interval [x , b].

In the ordinary calculus, the properties of the operators of integration and differentiation with respect to the laws of commutatively and additively of their (integer) exponents are well known. Using our notation, the (trivial) laws

$$J^m J^n = J^n J^m = J^{m+n}, D^m D^n = D^n D^m = D^{m+n} \qquad (2.1.23)$$

Where *m, n=0, 1, 2,….* Can be referred to as the law of exponents for the operators of integration and differentiation of integer order, respectively. Of course, for any positive integer order, the operators $D^{\propto}$ *and* J^{β} do not commute.

In the fractional calculus, the "law of exponents" is known to be generally true for the operators of fractional integration thanks to their semi group property

$$J^{\propto} J^{\beta} = J^{\beta+\propto}, \propto, \beta \geq 0 \qquad (2.1.24)$$

Which implies the commutative property

$J^{\propto} J^{\beta} = J^{\beta} J^{\propto}$ (2.1.25)

In general, the operator of fractional differentiation, $D^{\propto}$,dose not satisfy either the semi group property, or the (weaker) commutative property, that is why so-called *sequential derivatives* were introduced by Miller and Ross. See (Miller and Ross, 1993)

To show how the law of exponents does not necessarily hold for the standard fractional derivative, we provide two simple examples (with power functions) for which

(i) $D^{\propto} D^{\beta} f = D^{\beta} D^{\propto} f \neq D^{\propto+\beta} f$

(ii) $D^{\propto} D^{\beta} g \neq D^{\beta} D^{\propto} g = D^{\propto+\beta} g$

In the example (i), let us take $f(x) = x^{-1/2}$ *and* $\propto = \beta = 1/2$.Then, using Example 2.1.3, we get

$$D^{\frac{1}{2}} f(x) \equiv 0, \qquad D^{\frac{1}{2}} D^{\frac{1}{2}} f(x) \equiv 0 \text{ , but}$$

$$D^{\frac{1}{2}+\frac{1}{2}} f(x) = Df(x) = \frac{-x^{-3/2}}{2}$$

In the example (ii), let us take $g(x) = x^{1/2}$ *and* $\propto = 1/2, \beta = 3/2$. then, again using Example 2.1.3, we get

$$D^{\frac{1}{2}} g(x) \equiv \frac{\sqrt{\pi}}{2}, \qquad D^{\frac{3}{2}} g(x) \equiv 0 \text{ , but } D^{\frac{1}{2}} D^{\frac{3}{2}} g(x) \equiv 0$$

$$D^{\frac{3}{2}} D^{\frac{1}{2}} g(x) = \frac{-x^{3/2}}{4} \text{ and } D^{\frac{1}{2}+\frac{3}{2}} g(x) = D^{2} g(x) = \frac{-x^{3/2}}{4}$$

Theorem 2.1.4 (Diethelm, 2009) let f_1 *and* f_2 be two functions defined on *[a, b]* such that $D_{a^+}^{\propto} f_1$ and $D_{a^+}^{\propto} f_2$ exist almost everywhere. Moreover, let c_1 *and* $c_2 \in R$. *Then*

$$D_{a^+}^{\propto}(c_1 f_1 + c_2 f_2) = D_{a^+}^{\propto} c_1 f_1 + D_{a^+}^{\propto} c_2 f_2 \qquad (2.1.26)$$

We can apply this theorem for the operator $D_{b^-}^{\propto}$

2.1.5 Relations between Riemann-Liouville Integrals and Derivatives

Here, we investigate how the Riemann-Liouville differential and integral operators interact .

Theorem 2.1.5 (Miller and Ross, 1993) The left Riemann-Liouville fractional derivative of order α acts as a left inverse of the left Riemann-Liouville fractional integral of order α, i.e.

$$D_{a^+}^{\propto} J_{a^+}^{\propto} f(x) = f(x), \forall \propto > 0$$

That is $D_{a^+}^{\propto}$ is the left inverse of $J_{a^+}^{\propto}$

Proof: Using the definition of the left Riemann-Liouville fractional derivative, Theorem 2.1.5, and the Fundamental Theorem of Calculus (Theorem 2.1.1), we have

$$D_{a^+}^{\propto} J_{a^+}^{\propto} f(x) = D^n D_{a^+}^{\propto - n} D_{a^+}^{-\propto} f(x) = D^n J_{a^+}^{n} f(x) = f(x)$$

Theorem 2.1.6 (Miller and Ross, 1993) The right Riemann-Liouville fractional derivative of order α acts as a left inverse of the right Riemann-Liouville fractional integral of order α, i.e.

$$D_{b^-}^{\propto} J_{b^-}^{\propto} f(x) = f(x), \forall \propto > 0 \qquad (2.1.28)$$

Proof: Using the definition of the right Riemann-Liouville fractional derivative,, we write

$$D_{b^-}^{\propto} J_{b^-}^{\propto} f(x) = (-D)^n D_{b^-}^{\propto - n} D_{b^-}^{\propto} f(x) = (-D)^n J_{b^-}^{n} f(x) = f(x)$$

Of course, we cannot claim that $D_{a^+}^{\propto}$ is the right inverse. More precisely, we have the following situation.

Theorem 2.1.7 (Miller and Ross, 1993) let $\propto > 0, n = \lfloor \propto \rfloor + 1$. The following composition rule holds for the left Riemann-Liouville fractional derivative operators

$$J_{a^+}^{\propto} D_{a^+}^{\propto} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\propto -k-1}}{\Gamma(\propto - k)} \lim_{z \to a^+} D^{n-k-1} J_{a^+}^{n-\propto} f(z) \qquad (2.1.29)$$

Theorem 2.1.8 (Miller and Ross, 1993) let $\propto > 0$. the following composition rule holds for the right Riemann-Liouville fractional derivative operators

$$J_{b^-}^{\propto} D_{b^-}^{\propto} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(b-x)^{\propto -k-1}}{\Gamma(\propto - k)} \lim_{z \to b^-} D^{n-k-1} J_{b^-}^{n-\propto} f(z) \qquad (2.1.30)$$

Another important basic result in classical analysis is Taylor's theorem.

Theorem 2.1.9 (Diethelm, 2009) (Taylor's expansion)

Let $f \in C^n[a,b]. Then, for\ x \in [a,b],$

$$f(x) = \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} \mathrm{D}^k f(a) + J_{a^+}^{n} \mathrm{D}^n f(x) \qquad (2.1.31)$$

Based on the results derived so far we can find a fractional generalization of this statement.

Theorem 2.1.10 (Diethelm, 2009) (Fractional Taylor's expansion). Let $\propto > 0, n = \lfloor \propto \rfloor + 1$

Assume that f is such that $J_{a^+}^{n-\propto} f \in A^n[a,b]$. (by $A^n[a,b]$ we denote the set of functions with an absolutely continues (n-1) st derivative), we have

$$f(x) = \frac{(x-a)^{\propto - n}}{\Gamma(\propto - n + 1)} \lim_{z \to a^+} J_{a^+}^{n-\propto} f(z) + \sum_{k=1}^{n-1} \frac{(x-a)^{k+\propto -n}}{\Gamma(k + \propto - n + 1)} \lim_{z \to a^+} D_{a^+}^{k+\propto -n} f(z) + J_{a^+}^{\propto} D_{a^+}^{\propto} f(x) \qquad (2.1.32)$$

Note that in the case $\propto\in N\ we\ have\ n=\propto+1$,and the thus the limit outside the sum vanishes. We may thus retrieve the classical result (with n replaced by n-1)

2.1.6 The Grùnwald-Letnikov "Differintegral"(Diethelm, 2009)

In 1867, A.K. Grùnwald considered the definition of the fractional derivative, and in 1868, A.V. Letnikov, who did not know about Grùnwald results, suggested a similar approach. The Grùnwald-Letnikov approach to fractional differentiation is a generalization of the limit definition of the derivative

The derivative of a function *f* is defined as

$$D^1 f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad (2.1.33)$$

The formula for the derivative can be applied recursively to get higher order derivatives. For example, the second-order derivative would be:

$$D^2 f(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}$$

$$= \lim_{h_1\to 0} \frac{\dfrac{\lim_{h_2\to 0} f(x+h_1+h_2)-f(x+h_1)}{h_2} - \dfrac{\lim_{h_2\to 0} f(x+h_2)-f(x)}{h_2}}{h} \qquad (2.1.34)$$

By choosing the same value of$h$, i.e. $h=h_1=h_2$ the expression simplified to

$$D^2 f(x) = \lim_{h\to 0} \frac{f(x+2h)-2f(x+h)+f(x)}{h^2} \qquad (2.1.35)$$

Iterating this operation yields an expression for the nth derivatives of a function. As can be easily seen and proved by induction for any natural number n

$$D^n f(x) = \lim_{h \to 0} h^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x + (n-m)h) \tag{2.1.36}$$

Where

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!} \tag{2.1.37}$$

Or equivalently,

$$D^n f(x) = \lim_{h \to 0} h^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x - mh) \tag{2.1.38}$$

The case of n=0 can be included as well.

Viewing this expression, one asks immediately if it can be generalized to any non-integer, real or complex number n. There are some reasons that can make us think so:

1. The fact that for any natural number n, the calculation of the n-th derivative is given by an explicit formula (2.1.36) or (2.1.38).
2. That the generalization of the factorial by the Gamma function allows

   $$\binom{n}{m} = \frac{n!}{m!\,(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)} \tag{2.1.39}$$

   Which is also valid for non-integer values.
3. Tthe binomial formula.

   $$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m \tag{2.1.40}$$

   Which can be generalized to any complex number α by

   $$(a+b)^\alpha = \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha-n+1)} a^{\alpha-n} b^n \tag{2.1.41}$$

Which is convergent if $|b| < a$

There are some desirable properties that could be required to the fractional derivatives:

1. Existence and continuity of m times derivable functions, for any n whose modulus is equal or less than m.
2. For n=0, the result should be the function itself; if $n < 0$ integer values, it should be equal to ordinary integration regardless the integration constant.
3. Iterating should not give problems

   $$D^{\alpha+\beta}f(x) = D^{\alpha}D^{\beta}f(x) \qquad (2.1.42)$$

4. Linearity

   $$D^{\alpha}\left(c_1 f_1(x) + c_2 f_2(x)\right) = c_1 D^{\alpha} f_1(x) + c_2 D^{\alpha} f_2(x) \qquad (2.1.43)$$

5. Allowing Taylor's expansion in some other way. See (Munkhammar, 2004).
6. It's characteristic property should be preserved for the exponential function

   $$D^{\alpha}e^{x} = e^{x}$$

   Grünwald-Letnikov derivative or also named Grünwald-Letnikov differintegral, is a generalization of the derivative analogous to our generalization by the binomial formula

   $$D^{\alpha}f(x) = \lim_{h\to 0} h^{-\alpha}\sum_{n=0}^{\infty}(-1)^{n}\frac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha-n+1)}f(x-nh) \qquad (2.1.44)$$

   But it is based on the direct generalization of the equation (2.1.38).

   The idea behind, is that h should approach 0 as n approaches infinity,

   $$D^{n}f(x) = \lim_{h\to 0} h^{-n}\sum_{m=0}^{n}(-1)^{m}\binom{n}{m}f(x-mh)$$

   $$= \lim_{h\to 0} h^{-n}\sum_{m=0}^{n}(-1)^{m}\frac{n!}{m!\,(n-m)!}f(x-mh)$$

$$= \lim_{h\to 0} h^{-n} \sum_{m=0}^{\infty} (-1)^m \frac{n!}{m!\,(n-m)!} f(x-mh)$$

$$= \lim_{h\to 0} h^{-n} \sum_{m=0}^{h} (-1)^m \frac{n!}{m!\,\Gamma(n-m+1)} f(x-mh)$$

Being a negative infinity, the generalization is done so that any wished a less than x can be chosen.

The above equation can be generalized now to get

$$D^n f(x) = \lim_{h\to 0} h^{-n} \sum_{m=0}^{\frac{x-a}{h}} (-1)^m \frac{\Gamma(\alpha+1)}{m!\,\Gamma(\alpha-m+1)} f(x-mh) \qquad (2.1.45)$$

Or equivalently

$$D^n f(x) = \lim_{n\to\infty} \left(\frac{n}{x-a}\right)^{\alpha} \sum_{m=0}^{n} (-1)^m \frac{\Gamma(\alpha+1)}{m!\,\Gamma(\alpha-m+1)} f\left(x - m\left(\frac{x-a}{h}\right)\right) \qquad (2.1.46)$$

These generalizations have the inconvenience when α is a negative integer, since the Gamma function is infinite for negative integers and zero.

However, for negative α values, the following can be used.

$$\binom{-n}{m} = \frac{\prod_{l=-n-m+1}^{-n} l}{m!} = \frac{\prod_{l=-n-m+1}^{-n} l}{m!} = \frac{(-1)^m \prod_{l=n}^{n+m-1} l}{m!} = \frac{(-1)^m (m+m-1)!}{m!\,(n-1)!}$$
$$= \frac{(-1)^m \Gamma(n+m)}{m!\,\Gamma(n)} \qquad (2.1.47)$$

So that, for negative values of α, equations (2.1.45) and (2.1.46) can be changed to

$$J^{\alpha} f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{m=0}^{\frac{x-a}{h}} \frac{\Gamma(\alpha+\mathrm{m})}{m!\,\Gamma(\alpha)} f(x-mh) \qquad (2.1.48)$$

And

$$J^{\alpha} f(x) = \lim_{n\to\infty} \left(\frac{n}{x-a}\right)^{\alpha} \sum_{m=0}^{n} \frac{\Gamma(\alpha+m)}{m!\,\Gamma(\alpha)} f\left(x - m\left(\frac{x-a}{h}\right)\right) \qquad (2.1.49)$$

The reason why $x - a$ is considered in the above equations instead of a single positive number, can be seen considering the case of$\alpha = -1$, which must give the same result when integrating the function

$$\int_{a}^{x} f(t)dt = D^{-1} f(x) = \lim_{\substack{h\to 0 \\ n\to\infty}} h \sum_{m=0}^{n} \frac{\Gamma(m+1)}{m!\,\Gamma(1)} f(x - mh)$$

$$= \lim_{\substack{h\to 0 \\ n\to\infty}} \sum_{m=0}^{n} h\, f(x - mh) = \lim_{\substack{h\to 0 \\ n\to\infty}} \int_{0}^{nh} f(x-t)dt = \lim_{\substack{h\to 0 \\ n\to\infty}} \int_{x-nh}^{x} f(t)dt$$

$$\gg a = x - nh \gg n = \frac{x-a}{h} \quad , h = \frac{x-a}{n}$$

Yes, in fractional derivatives, limits must be considered as they must be in integrals; limits only vanish with integer order derivatives. This also means that fractional derivatives are nonlocal, which may be the reason that makes this kind of derivatives less useful in describing nature.

In the case of the derivatives generalized by the binomial formula

$$D^{\alpha} f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{n!\,\Gamma(\alpha - n + 1)} f(x - nh) \qquad (2.1.50)$$

Since n goes to infinity regardless of h, *x-a* must be infinity, so that the derivative defined in

$$D^{\alpha} f(x) = \lim_{h\to 0} \left(\frac{d_n - 1}{h}\right)^{\alpha} f(x) \qquad (2.1.51)$$

And

$$D^{\propto} f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha-n+1)} f(x-nh) \qquad (2.1.52)$$

Is equivalent to the Grȕnwald-Letnikov derivative with a lower limit of negative infinity .

Definition 2.1.8 [Grȕnwald-Letnikov fractional derivative] Let $f(x)$ be defined on the interval *(a, b)* and $\propto > 0$ is defined to be

$$D^{\propto} f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha-n+1)} f(x-nh) \qquad (2.1.53)$$

Example 2.1.6 Let $f(x) = e^{\beta x}$. Then [by direct generalization of (2.1.36)]

$$D^{\propto} e^{\beta x} = \lim_{h\to 0} h^{-\alpha} \sum_{n=0}^{\alpha} (-1)^n \binom{\alpha}{n} e^{\beta(x+(\alpha-n)h)}$$

$$= e^{\beta x} \lim_{h\to 0} h^{-\alpha} \sum_{n=0}^{\alpha} (-1)^n \binom{\alpha}{n} (e^{\beta h})^{(\alpha-n)}$$

$$= e^{\beta x} \lim_{h\to 0} h^{-\alpha} (e^{\beta h} - 1)^{\alpha} = \beta^{\alpha} e^{\beta x}$$

Example 2.1.7 Let $f(x) = x^{\beta}$.Then

$$D^1 x^{\beta} = \beta x^{\beta-1} \gg D^n x^{\beta} = x^{\beta-n} \prod_{m=0}^{n-1} (\beta - m) = \frac{\beta!}{(\beta-n)!} x^{\beta-n}$$

Can be easily generalized to non-integer order derivatives

$$D^{\propto} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}$$

Which can be applied to any function that can be expanded on powers of x

$$f(x) = \sum_{n=-\infty}^{\infty} c_n x^n = D^{\propto} f(x) = \sum_{n=-\infty}^{\infty} c_n D^{\propto} x^n = \sum_{n=-\infty}^{\infty} c_n \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

Theorem 2.1.11 (Miller and Ross, 1993) the left Riemann-Liouville and Grünwald-Letnikov fractional integral operators are equivalent, provided that $J_{a^+}^{\propto} f(x)$ is finite for all values of x

Proof: Using the definiyion of Grünwald-Letnikov fractional integral, we have

$$J_{a^+}^{\propto} f(x) = \lim_{n\to\infty} \left(\frac{x-a}{n}\right)^{\alpha} \sum_{k=0}^{n} \frac{\Gamma(\alpha+k)}{\Gamma(k+1)\Gamma(\alpha)} f\left(x - k\left(\frac{x-a}{n}\right)\right) \qquad (2.1.54)$$

Writing this in terms of a Riemann sum over the interval [a, x], we have

$$J_{a^+}^{\propto} f(x) = \frac{1}{\Gamma(\alpha)} \lim_{n\to\infty} \sum_{k=0}^{n} \beta_k \left(\frac{k(x-a)}{n}\right)^{\alpha-1} f\left(x - \frac{k(x-a)}{n}\right)\left(\frac{x-a}{n}\right),$$

Where $\beta_k = \frac{\Gamma(\alpha+k)k^{1-\propto}}{\Gamma(k+1)} = \frac{\Gamma(\alpha+k)k^{-\propto}}{\Gamma(k)}$

Note that if the β_k term is absent, then (2.1.54) implies

$$J_{a^+}^{\propto} f(x) = \frac{1}{\Gamma(\propto)} \int_0^{x-a} \xi^{\alpha-1} f(x-\xi)\, d\xi = J_{a^+}^{\propto} f(x) \ (in\ Riemann - Liouville\ sense)$$

We obtain the equivalence via a result of Letnikov, which states that if $\lim_{k\to\infty} \beta_k =, \lim_{n\to\infty} \alpha_{n,k} = 0\ , \forall k \in N$

$$\sum_{k=1}^{n-1} |\propto_{n,k}| < k, \forall n \in N\ and\ \lim_{n\to\infty} \sum_{k=1}^{n-1} \propto_{n,k} = A$$

Then $\lim_{n\to\infty} \sum_{k=1}^{n-1} \beta_k \propto_{n,k} = A\ also.$

Therefore, setting

$$\propto_{n,k} = \left(\frac{k(x-a)}{n}\right)^{\alpha-1} f\left(x - \frac{k(x-a)}{n}\right)\left(\frac{x-a}{n}\right)$$

The stated result follows.

2.1.7 Caputo's Approach

The Riemann-Liouville derivatives have certain disadvantages when trying to model real-world phenomena with fractional differential equations. The Caputo approach is a modified concept of fractional derivative (Diethelm, 2009)

The Caputo approach to fractional differentiation is the same as the approach of Riemann and Liouville, except that the order in which the integration and differentiation are done is reversed. The Caputo derivative dose not satisfy the inverse property as in Theorem 2.1.5.However, an advantage of the Caputo derivatives is that boundary conditions depend upon integer order derivatives values (Miller and Ross, 1993)

Definition 2.1.9 (Miller and Ross, 1993) [Caputo Fractional Derivatives] Let f be defined on the interval *(a, b)* and $\propto > 0$,n be the smallest integer greater than $\alpha(n-1 \leq \propto \leq n)$. Then the Caputo fractional derivative of order α is defined to be

$$D_{*a}^{\propto} f(t) := f(x) = \begin{cases} \frac{1}{\Gamma(n-\propto)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt, & n-1 \leq \propto < n; \\ \frac{d^n}{dx^n} f(x) \quad , & \propto = n \end{cases} \quad (2.1.55)$$

2.2 Spline Interpolation Function

2.2.1 Introduction

Interpolation is used to estimate the value of a function between known data points without knowing the actual function. Interpolation methods can be divided into two main categories 1 and 2:

1) **Global interpolation**: These methods rely on a constructing single equation that fits all the data points. This equation is usually a high degree polynomial equation. Although these methods result in smooth curves, they are usually not well suited for engineering applications, as they are prone to severe oscillation and overshoot at intermediate points.

2) **Piecewise interpolation**: These methods rely on constructing a polynomial of low degree between each pair of known data points. If a first degree polynomial is used, it is called linear interpolation. For second and third degree polynomials, it is called quadratic and cubic splines respectively. The higher the degree of the spline, the smoother the curve. Splines of degree m, will have continuous derivatives up to degree m-1 at the data points.

We have been interpolating the data points (x_0,y_0) to (x_n,y_n) with a polynomial of degree at most n. We now focus our attention on a deferent strategy: splines.

A function S(x) is called a spline of degree k if

(i) The domain of S is an interval *[a, b]*.

(ii) $S; S', \ldots\ldots, S^{(k-1)}$ are continuous on *[a, b]*.

(iii) There are points xi such that $a = x0 < x1 < \ldots. < xn = b$ and such that S is a polynomial of degree at most k on each $[x_i , x_{i+1}]$

In other words, splines are piecewise polynomial functions whose derivatives obey certain continuity constraints.

When k = 1, the splines are called linear splines. When k = 2, the splines are called quadratic splines, and when k = 3, the splines are called cubic splines.

2.2.2 Linear Spline Interpolation (Meir & Sharma, 1973)

Linear interpolation result in straight line between each pair of points and all derivatives are discontinuous at the data points. As it never overshoots or oscillates, it is frequently used in chemical engineering despite the fact that the curves are not smooth.

We seek a linear spline function *S(x)* such that *S (xi) = yi* for $0 \le i \le n$

Let

$$S(x) = \begin{cases} S_0(x), & x_0 \le x \le x_1 \\ S_1(x), & x_1 \le x \le x_2 \\ \vdots & \vdots \\ S_{n-1}(x), & x_{n-1} \le x \le x_n \end{cases}$$

where each $S_i(x)$ is linear

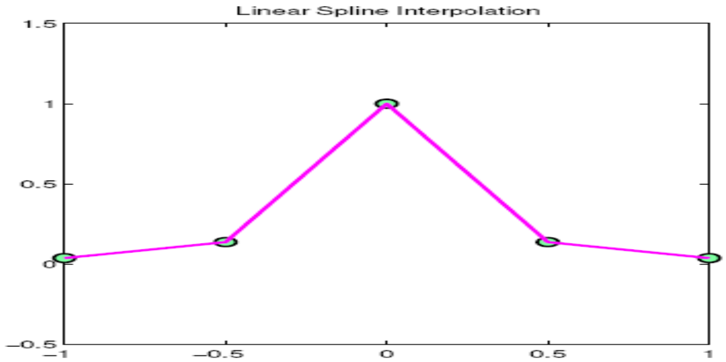


Figure 2.1: Linear Spline Interpolation

Example 2.2.1 Consider the problem of finding a linear spline for

$$f(x) = e^{-x^2}$$

with the following

$$(x_0, f(x_0)) = (x_0, y_0) = \left(-1,\ e^{-1}\right)$$
$$(x_1, f(x_1)) = (x_1, y_1) = \left(-\frac{1}{2},\ e^{-\frac{1}{4}}\right)$$

$$(x_2, f(x_2)) = (x_2, y_2) = (0,\ 1)$$
$$(x_3, f(x_3)) = (x_3, y_3) = \left(\frac{1}{2},\ e^{-\frac{1}{4}}\right)$$
$$(x_4, f(x_4)) = (x_4, y_4) = \left(1,\ e^{-1}\right)$$

After directly applying the spline formula, we get the following spline:

$$S(x) = \begin{cases} e^{-1} + 2(e^{-\frac{1}{4}} - e^{-1})(x+1) & x \in [-1, -\frac{1}{2}] \\ e^{-\frac{1}{4}} + 2(1 - e^{-\frac{1}{4}})(x + \frac{1}{2}) & x \in [-\frac{1}{2}, 0] \\ 1 + 2(e^{-\frac{1}{4}} - 1)x & x \in [0, \frac{1}{2}] \\ e^{-\frac{1}{4}} + 2(e^{-1} - e^{-\frac{1}{4}})(x - \frac{1}{2}) & x \in [\frac{1}{2}, 1] \end{cases}$$

The spline function (lines) and the function it is approximating (dashes) are graphed below:

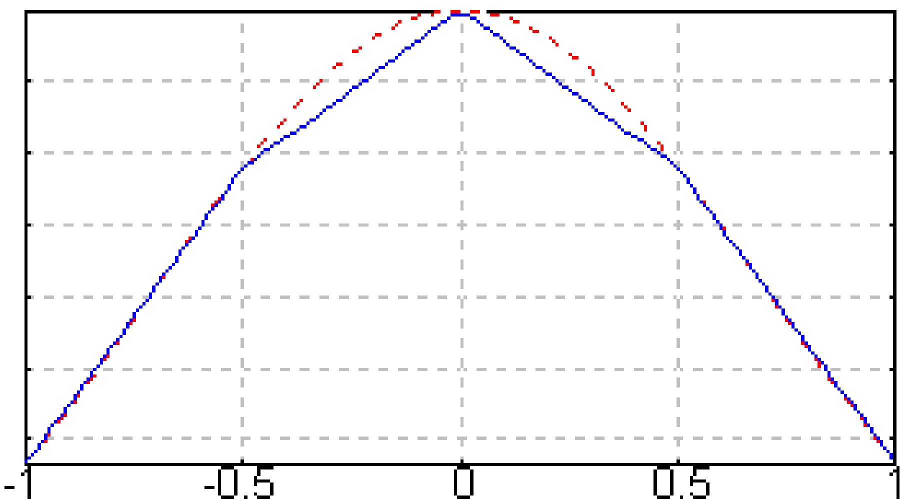


Figure 2.2: Linear Spline Interpolation for $f(x) = e^{-x^2}$

A disadvantage of linear function approximation is that there is likely no differentiability at the endpoint of the subintervals, which, in a geometrical context, means that the interpolating function is not smooth. Often it is clear from physical conditions that smoothness is required, so the approximating function must be continuously differentiable.

2.2.3 Quadratic Spline Interpolation (Meir & Sharma, 1973)

Given the data points $(x_0, y_0), \dots \dots \dots \dots \dots \dots, (x_n, y_n)$.The objective in quadratic spline interpolation is to derive a 2-order polynomial for each interval between data points. The polynomial for each interval can be represent generally as

$\boldsymbol{f_i(x) = a_i x^2 + b_i x + c_i}$

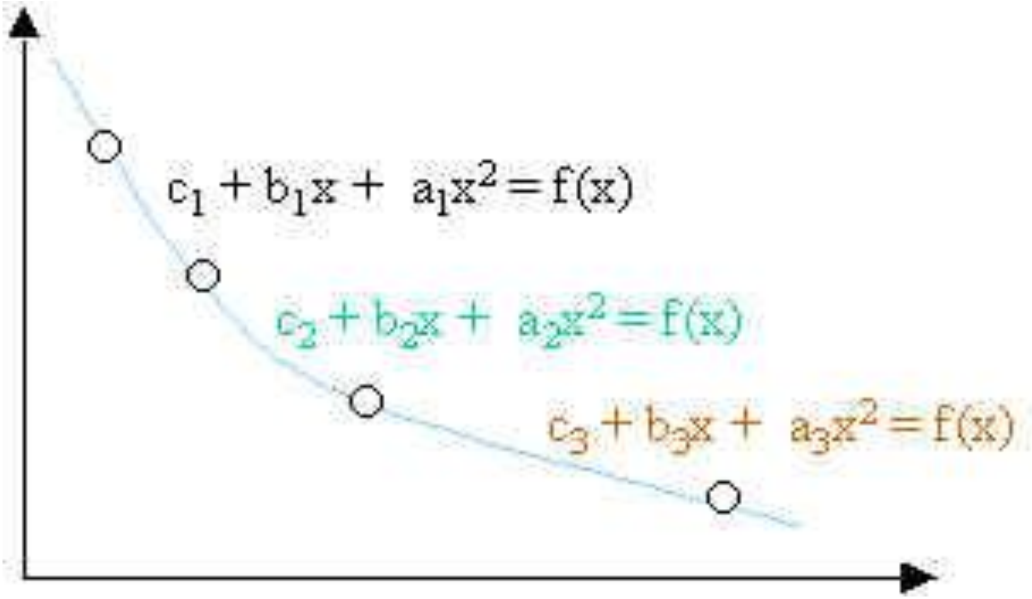


Figure 2.3: Quadratic Spline Interpolation

So for (n+1) data points there exist n intervals and 3n unknowns (a_i, b_i, c_i) and 3n equations are required to evaluate. Each spline goes through two consecutive data point. The function values of adjacent polynomial must be equal at the interior knots. This condition can be represented

$a_i {x_{i-1}}^2 + b_i x_{i-1} + c_i = y_{i-1}$

$a_i {x_i}^2 + b_i x_i + c_i = y_i$ (2.2.1)

For i=1,…, n for 2n equations so 3n-2n=n equations are needed. The first and last function must pass through end point. This adds two additional conditions:

$a_1 {x_0}^2 + b_1 x_0 + c_1 = f(x_0)$

$a_n {x_n}^2 + b_n x_n + c_n = f(x_n)$ (2.2.2)

The derivative at the interior nodes must be equal that means each spline has same slop at the interior data point (knot).the condition can be represented generally

$$\frac{d}{dx}(a_i x^2 + b_i x + c_i)|(x = x_i) = \frac{d}{dx}(a_{i+1} x^2 + b_{i+1} x + c_{i+1})|(x = x_i)$$

$2a_i x_i + b_i = 2a_{i+1} x_i + b_{i+1}$ *for i=1,….., n-1*

We need also to take the condition$a_1 = 0$.

So, the simplest type of differentiable piecewise-polynomial function on an entire interval $[x_0, x_n]$ is the function obtained by fitting one quadratic polynomial between each successive pair of nodes. This is one by constructing one quadratic on $[x_0, x_1]$agreeing with the function at x_0 and x_1, another quadratic on $[x_1, x_2]$ agreeing with the function at x_1 and x_2 and so on. Since a general quadratic polynomial has three arbitrary constants, the constant term is the coefficient of x and the coefficient of x^2 and only two conditions are required to fit the data at the endpoints of each subinterval, flexibility exists that permits the quadratic to be chosen so that the interpolate has a continuous derivative on $[x_0, x_n]$.

The difficulty arises when there is a need to specify conditions about the derivative of the interpolation at the endpointsx_0 *and* x_n. There are not a sufficient number of constants to ensure that the conditions will be satisfied.

2.2.4 Cubic Spline Interpolation (Meir & Sharma, 1973)

Real world numerical data is usually difficult to analyze. Any function which would effectively correlate the data would be difficult to obtain and highly unwieldy. To this end, the idea of the cubic spline was developed. Using this process, a series of unique cubic polynomials are fitted between each of the data points, with the stipulation that the curve obtained be continuous and appear smooth. These cubic splines can then be used to determine rates of change and cumulative change over an interval.

The fundamental idea behind cubic spline interpolation is based on the engineer's tool used to draw smooth curves through a number of points. This spline consists of weights attached to a flat surface at the points to be connected. A flexible strip is then bent across each of these weights, resulting in a pleasingly smooth curve.

The mathematical spline is similar in principle. The points, in this case, are numerical

data. The weights are the coefficients on the cubic polynomials used to interpolate the data. These coefficients ’bend’ the line so that it passes through each of the data points without any erratic behavior or breaks in continuity.

Cubic splines are frequently recommended. They are generally well behaved and continuous up to the second order derivative at the data points. Even though cubic splines are less prone to oscillation or overshoot than global polynomial equations, they do not prevent it. Thus, the use of cubic splines in chemical engineering is limited to applications where oscillation and overshoot are acceptable or desirable.

Process The essential idea is to fit a piecewise function of the form:

$$S(x) = \begin{cases} S_1(x), \ x_1 \le x \le x_2 \\ S_2(x), \ x_2 \le x \le x_3 \\ . \qquad\qquad\qquad . \\ . \qquad\qquad\qquad . \\ . \qquad\qquad\qquad . \\ S_{n-1}(x), \ x_{n-1} \le x \le x_n \end{cases} \qquad (2.2.3)$$

where *Si* is a third degree polynomial defined by

$$S_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \qquad (2.2.4)$$

for $i = 1, 2, ..., n - 1$.

The first and second derivatives of these $n - 1$ equations are fundamental to this process, and they are

$$S'_i(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i \qquad (2.2.5)$$

$$S''_i(x) = 6a_i(x - x_i) + 2b_i \qquad (2.2.6)$$

for $i = 1,2,\ldots, n-1$.

The Four Properties of Cubic Splines

Our spline will need to conform to the following stipulations.

1. The piecewise function *S(x)* will interpolate all data points.

2. S(x) will be continuous on the interval $[x_1, x_n]$

3. *S’(x)* will be continuous on the interval $[x_1, x_n]$

4. *S''(x)* will be continuous on the interval $[x_1,x_n]$

Since the piecewiece function *S*(x) will interpolate all of the data points, we can conclude that

$$S(x_i) = y_i \qquad (2.2.7)$$

for i=1,…… …., n-1, since $x_i \in [x_i, x_{i+1}], S(x_i) = S_i(x_i)$, and we can use equation (2.2.4) to produce

$$y_i \;=\; S_i(x_i)$$

$$y_i \;= a_i(x_i - x_i)^3 + b_i(x_i - x_i)^2 + c_i(x_i - x_i) + d_i$$

$$y_i = d_i$$

$$\text{for each } i = 1,2,\ldots\ldots, n-1$$

Since the curve S(x) must be continuous across it's entire interval. It can be concluded that each sub-function must join at the data points, so

$$S_i(x_i) = S_{i-1}(x_i) \qquad (2.2.8)$$

$$\text{for } \; i = 2,\ldots\ldots, n$$

From equation (2.2.4)

$$S_i(x_i) = d_i$$

and

$$S_{i-1}(x_i) = a_{i-1}(x_i - x_{i-1})^3 + b_{i-1}(x_i - x_{i-1})^2 + c_{i-1}(x_i - x_{i-1}) + d_{i-1} \qquad (2.2.9)$$

So

$$d_i \;= a_{i-1}(x_i - x_{i-1})^3 + b_{i-1}(x_i - x_{i-1})^2 + c_{i-1}(x_i - x_{i-1}) + d_{i-1} \qquad (2.2.10)$$

for i=2, 3,…., n-1

Now letting $h = x_i - x_{i-1}$ in equation (2.2.9), we will have

$$d_i \;= a_{i-1}h^3 + b_{i-1}h^2 + c_{i-1}h + d_{i-1} \qquad (2.2.11)$$

for i=2, 3,…., n-1

Also, to make the curve smooth across the interval, the derivatives must be equal at the data point, so that

$$S'_i(x_i) = S'_{i-1}(x_i) \qquad (2.2.12)$$

However, by equation (2.2.5)

$$S'_i(x_i) = c_i$$

And

$$S'_{i-1}(x_i) = 3a_{i-1}(x_i - x_{i-1})^2 + 2b_{i-1}(x_i - x_{i-1}) + c_{i-1}$$

$$c_i = 3a_{i-1}(x_i - x_{i-1})^2 + 2b_{i-1}(x_i - x_{i-1}) + c_{i-1} \qquad (2.2.13)$$

Again by letting $h = x_i - x_{i-1}$, we arrive at

$$c_i = 3a_{i-1}h^2 + 2b_{i-1}h + c_{i-1} \qquad (2.2.14)$$

for i=2, 3,…., n-1

From equation (2.2.6)

$$S''_i(x) = 6a_i(x - x_i) + 2b_i$$

So

$$S''_i(x_i) = 6a_i(x_i - x_i) + 2b_i \qquad (2.2.15)$$

$$S''_i(x_i) = 2b_i$$

for i=2,3,….,n-2

Lastly since $S''_i(x)$ has to be continuous across the interval, $S''_i(x_i) = S''_{i+1}(x_i)$ *for i=1, 2,..., n-1*

This and equation (2.2.15) lead us to the equations

$$S''_i(x_{i+1}) = 6a_i(x_{i+1} - x_i) + 2b_i \qquad (2.2.16)$$

$$S''_{i+1}(x_{i+1}) = 6a_i(x_{i+1} - x_i) + 2b_i \qquad (2.2.17)$$

And $h = x_i - x_{i-1}$ and using the conclusion from equations (2.2.15) and (2.2.17)

$$S''_{i+1}(x_{i+1}) = 6a_i(x_{i+1} - x_i) + 2b_i \qquad (2.2.18)$$

$$2b_{i+1} = 6a_i h + 2b_i \qquad (2.2.19)$$

These equation can be simplified by substituting M_i for $S''_i(x_i)$ and expressing the above equation in terms of M_i and y_i .This makes the determination of the weights a_i, b_i, c_i and d_i a much easier task.

Each b_i can be represented by

$$S''_i(x_i) = 2b_i \qquad (2.2.20)$$

$$M_i = 2b_i$$

$$b_i = \frac{M_i}{2}$$

And d_i has already been determined to be

$$d_i = y_i \qquad (2.2.21)$$

Similarly, using a_i can be re-written as

$$2b_{i+1} = 6a_i h + 2b_i$$

$$6a_i h = 2b_{i+1} - 2b_i$$

$$a_i = \frac{2b_{i+1} - 2b_i}{6h}$$

$$a_i = \frac{2(\frac{M_{i+1}}{2}) - 2(\frac{M_i}{2})}{6h}$$

$$a_i = \frac{M_{i+1} - M_i}{6h} \qquad (2.2.22)$$

And c_i can be re-written as

$$d_{i+1} = a_i h^3 + b_i h^2 + c_i h + d_i$$

$$c_i h = -a_i h^3 - b_i h^2 - d_i + d_{i+1}$$

$$c_i = \frac{-a_i h^3 - b_i h^2 - d_i + d_{i+1}}{h}$$

$$c_i = \frac{-a_i h^3 - b_i h^2}{h} + \frac{-d_i + d_{i+1}}{h}$$

$$c_i = (-a_i h^2 - b_i h) - \frac{d_i - d_{i+1}}{h}$$

$$c_i = -\left(\frac{M_{i+1} - M_i}{6h} h^2 + \frac{M_i}{2} h\right) - \frac{y_i - y_{i+1}}{h}$$

$$c_i = \frac{y_{i+1} - y_i}{h} - \left(\frac{M_{i+1} - M_i}{6} h + \frac{3M_i}{6} h\right)$$

$$c_i = \frac{y_{i+1} - y_i}{h} - \left(\frac{M_{i+1} - M_i + 3M_i}{6}\right) h$$

$$c_i = \frac{y_{i+1} - y_i}{h} - \left(\frac{M_{i+1} - 2M_i}{6}\right) h \qquad (2.2.23)$$

So, now we have our equations for determing the weights for our n-1 equations

$$a_i = \frac{M_{i+1} - M_i}{6h}$$

$$b_i = \frac{M_i}{2}$$

$$c_i = \frac{y_{i+1} - y_i}{h} - \left(\frac{M_{i+1} - 2M_i}{6}\right) h$$

$$d_i = y_i$$

These systems can be handled more conveniently by putting them into matrix form as follows

$$c_{i+1} = 3a_i h^2 + 2b_i h + c_i$$

$$3\left(\frac{M_{i+1} - M_i}{6h}\right) h^2 + 2\left(\frac{M_i}{2}\right) h + \frac{y_{i+1} - y_i}{h} - \left(\frac{M_{i+1} - 2M_i}{6}\right) h = \frac{y_{i+2} - y_{i+1}}{h} - \left(\frac{M_{i+2} - 2M_{i+1}}{6}\right) h$$

$$3\left(\frac{M_{i+1} - M_i}{6h}\right) h^2 + 2\left(\frac{M_i}{2}\right) h - \left(\frac{M_{i+1} - 2M_i}{6}\right) h + \left(\frac{M_{i+2} - 2M_{i+1}}{6}\right) h = -\left(\frac{y_{i+1} - y_i}{h}\right) + \frac{y_{i+2} - y_{i+1}}{h}$$

$$h\left(\frac{3M_{i+1} - 3M_i}{6} + 6\frac{M_i}{6} - \left(\frac{M_{i+1} - 2M_i}{6}\right) + \left(\frac{M_{i+2} + 2M_{i+1}}{6}\right)\right) = \frac{y_i - 2y_{i+1} + y_{i+2}}{h}$$

$$\frac{h}{6}(M_i + 4M_{i+1} + M_{i+2}) = \frac{y_i - 2y_{i+1} + y_{i+2}}{h}$$

$$(M_i + 4M_{i+1} + M_{i+2}) = 6(\frac{y_i - 2y_{i+1} + y_{i+2}}{h^2})$$

for i=1, 2,….., n-1

Which leads to the matrix equation

$$\begin{bmatrix} 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \vdots \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

Note that this system has *n* - 2 rows and *n* columns, and is therefore under-determined. In order to generate a unique cubic spline, two other conditions must be imposed upon the system

Types of Cubic Splines

We have three types of cubic spline Natural splines, Parabolic Runout Spline and Cubic Runout Spline

1) **Natural splines**

This first spline type includes the stipulation that the second derivative be equal to zero at the endpoints.

$$M_1 = M_n = 0 \qquad (2.2.26)$$

This results in the spline extending as a line outside the endpoints. The matrix for determining the M_1 - M_n values can be adapted accordingly.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ M_2 \\ M_3 \\ M_4 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \\ 0 \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \vdots \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

(2.2.27)

For reasons of convenience, the first and last columns of this matrix can be eliminated, as they correspond to the M_1 and M_n values, which are both 0.

$$\begin{bmatrix} 4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \vdots \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

(2.2.28)

This results in an n -2 by n -2 matrixes, which will determine the remaining solutions for M_2 Through M_{n-1}. The spline is now unique.

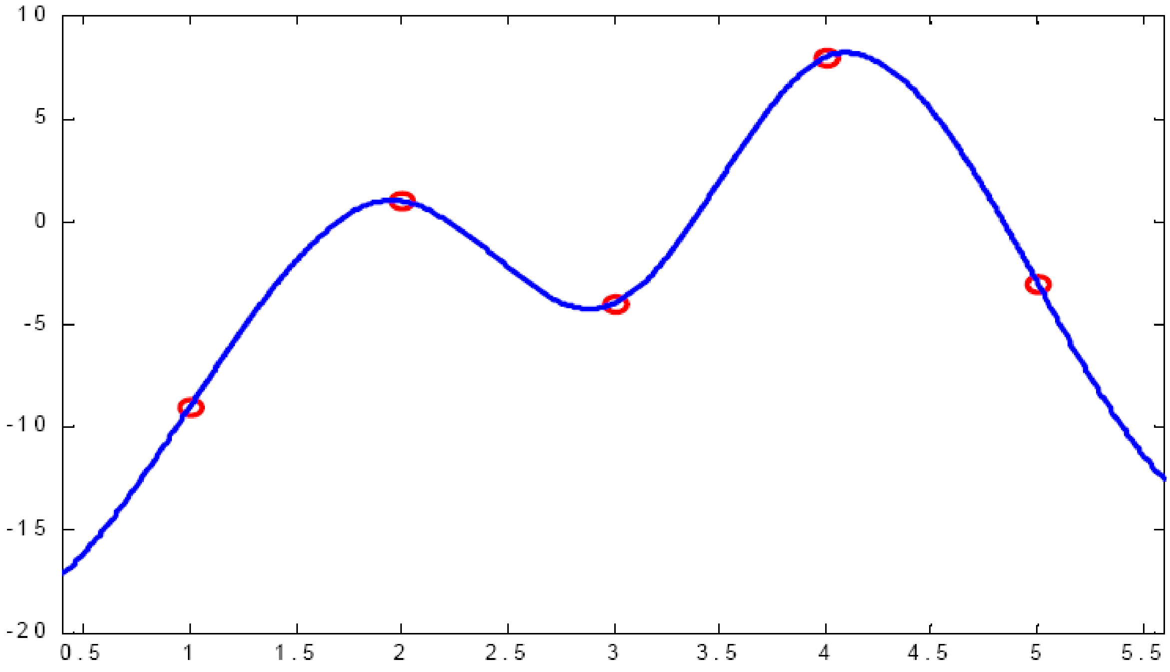


Figure 2.4: Natural interpolating curve

2) **Parabolic Runout Spline**

The parabolic spline imposes the condition that the second derivative at the endpoints, M_1 and M_n, be equal to M_2 and M_{n-1} respectively.

$$M_1 = M_2$$

$$M_n = M_{n-1} \qquad (2.2.29)$$

The result of this condition is the curve becomes a parabolic curve at the endpoint. This type of cubic spline is useful for periodic and exponential data.

The matrix equation for this type of spline is

$$\begin{bmatrix} 5 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \vdots \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

(2.2.30)

We can now determine the values for M_2 through M_{n-1}, with the values for M_1 and M_n already determined.

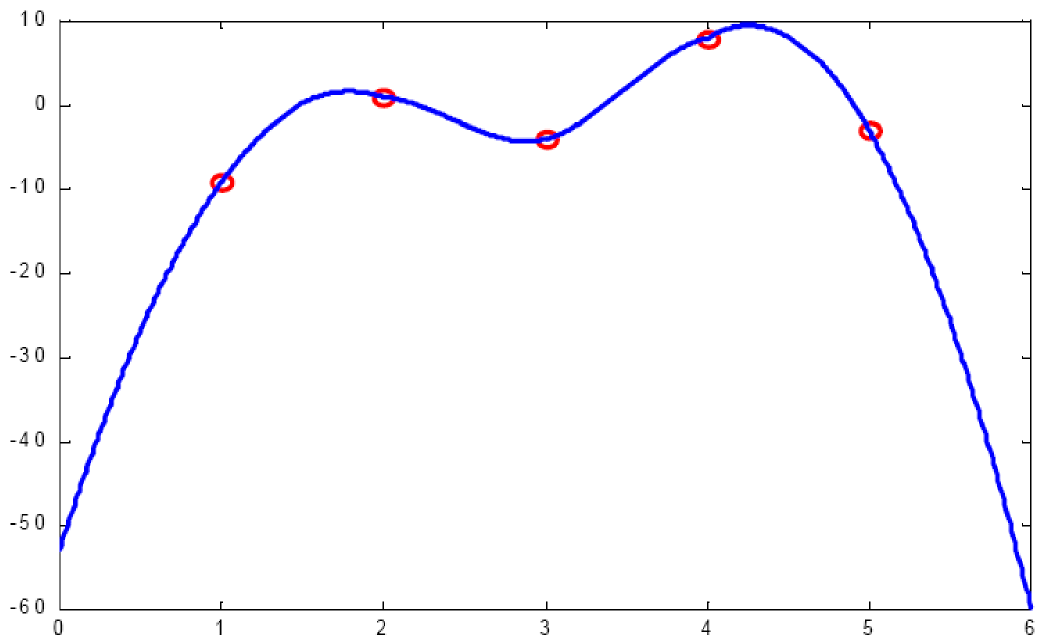


Figure 2. 5 : Parabolic Runout curve

Note that the endpoint behavior is a bit more extreme than with the natural spline option.

3) **Cubic Runout Spline**

This last type of spline has the most extreme endpoint behavior. It assigns M_1 to be $2M_2$ - M_3 and M_n to be $2M_{n-1} - M_{n-2}$. This causes the curve to degrade to a single cubic curve over the last two intervals, rather than two separate functions.

The matrix equation for this type is

$$\begin{bmatrix} 6 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \vdots \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

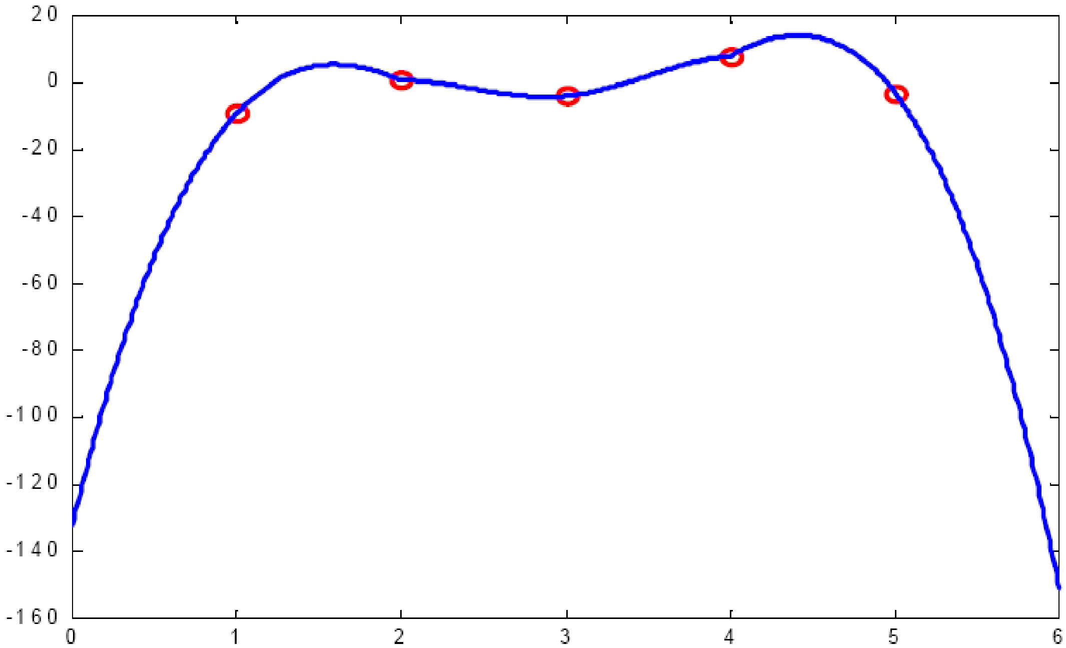


Figure 2.6: Cubic Runout curve

There are many other types of interpolating spline curves, such as the Periodic Spline and the Clamped Spline. The three discussed in this work are simply the ones which we have chosen to examine; they are not intrinsically superior to, or more widely used than these other types of splines.

Finally, Cubic spline interpolation is a powerful data analysis tool. Splines correlate data efficiently and effectively, no matter how random the data may seem. Once the algorithm for spline generation is produced, interpolating data with a spline becomes an easy task.

Chapter Three
Numerical Solution of Riccati Fractional Differential Equations Using Polynomial Spline Functions

3.1 Introduction

In the last few years there has been increasing interest in the use of various types of spline function in the numerical treatment of ordinary differential equation. Recently, fractional order differential equations have found interesting applications in the area of mathematical biology and due to the relation of such equations with memory that is inherit in corresponding biological systems. Also, a mathematical model of numerous engineering and physical phenomena involves either ordinary or partial differential equations of fractional order.

The analysis of fractional differential equation (existence, uniqueness, and stability) of the Form

$$y^{(\alpha)}(x) = f\big(x, y(x)\big), \qquad y(0) = y_0 \qquad (3.1.1)$$

is studied by (Diethelm & Ford, 2003). A number of approximate solutions of the initial value problem (3.1.1) have been proposed in the literature, where the Adams-Bashforth-Moulton method is introduced in (Diethelm, 1999) . An alternative method is the backward differentiation formula, presented in (Diethelm, 1997) where the idea of this method is based on discretizing the differential operator in the fractional differential equation (3.1.1) by certain finite difference. The main result of (Diethelm,1997) was that under suitable assumption we can expect an $O(h^{\alpha-2})$ convergence behavior. In (Ramadan, 2005) an improvement of the performance of the method presented in (Diethelm, 1997) is achieved by applying Extrapolation principles.(Diethelm, et al., 2006) is considered a fast algorithm for the numerical solution of initial value problems of the form (3.1.1) in the sense of Caputo identify and discuss potential problems in the development of generally applicable schemes. More recently, Lagrange multiplier method and

the homotopy perturbation method are used to solve numerically multi-order fractional differential equation .

Micula, et al. (1987) considered the problem

$$y'(x) = f_1(x,y,z) \qquad y(x_0) = y_0$$

$$z'(x) = f_2(x,y,z) \qquad z(x_0) = z_0 \qquad (3.1.2)$$

Where $f_1, f_2 \in C^r[0,1] \times R \times R, (x,,x) \in [0,1] \times R \times R$. They assumed that the function $f_i^{(q)}, i = 1,2 \; and \; q = 0,1,2,\dots,r$ satisfy the Lipschitz condition of the form:

$$\left|f_i^{(q)}(x,y_1,z_1) - f_i^{(q)}(x,y_2,z_2)\right| \leq L_i\{|y_1 - z_1| + |y_2 - z_2|\},$$

With constant $L_i \; for \; all \; (x,y_i,z_i) \in [a,b] \times R \times R$

And an investigations to the extension of the spline functions form defined in (Dirthelm, et al., 2006) for approximating the solution of system of ordinary differential equations, namely, for the system (3.2) with unique solution $y = y(x), z = z(x)$ is also considered. The splin functions $S_\Delta(x) and \widetilde{S_\Delta}(x)$ to approximate $y = y(x)$, $z = z(x)$ are defined in polynomial form as:

$$S_\Delta(x) = S_k(x) = S_{k-1}(x_k) + \sum_{i=0}^{r} f_1^{(i)}\left(x_k, S_{k-1}(x_k), \widetilde{S_{k-1}}(x_k)\right) \cdot \frac{(x - x_k)^{i+1}}{(i+1)!}$$

$$\widetilde{S_\Delta}(x) = \widetilde{S_k}(x) = \widetilde{S_{k-1}}(x_k) + \sum_{i=0}^{r} f_2^{(i)}\left(x_k, S_{k-1}(x_k), \widetilde{S_{k-1}}(x_k)\right) \cdot \frac{(x - x_k)^{i+1}}{(i+1)!}$$

$$x \in [x_k, x_{k-1}], k = 0,1,\dots,n-1, S_{-1}(x_0) = y_0, \widetilde{S_{-1}}(x_0) = z_0$$

Ramadan introduced in (Ramadan, 2005) the solution of the first order delay differential equation of the form:

$$y'(x) = f\left(x, y(x), y(g(x))\right) \qquad a \leq x \leq b$$

$$y(a) = y_0, y(x) = \varphi(x), x \in [a^*, a], a^* < 0, a^* = \inf\{g(x) : x \in [a,b]\}$$

Using the spline function of the polynomial form which defined as

$$S_\Delta(x) = S_k(x) = S_{k-1}(x_k) + \sum_{i=0}^{r} M_k^{(i)} \frac{(x - x_k)^{i+1}}{(i+1)!}$$

Where $M_k^{(i)} = f^{(i)}\left(x_k, S_{k-1}(x_k), S_{k-1}(g(x_k))\right)$ with $S_{-1}(x_0) = y_0, S_{-1}(g(x_0)) = \varphi(g(x_0))$

In (Ramadan, 2005) he considered the system of the initial value problem

$$y'''(x) = f_1(x, y, y', z, z') \qquad y^{(i)}(x_0) = y_0^{(i)}$$

$$z'''(x) = f_2(x, y, y', z, z') \qquad z^{(i)}(x_0) = z_0^{(i)}$$

Where $f_1, f_2 \in C^r([0,1] \times R^{(4)}), i = 0,1,2$. A method is presented which uses polynomial splines to approximate the solutions of the system.

Our purpose is to extend and generalize the polynomial spline functions used in the case of (Ramadan, 2005) and (Ramadan, et al., 2011), to approximate the solution of the fractional ordinary differential equation. The fractional derivative was defined in Caputo sence.This method is very powerful and efficient technique in finding numerical solutions for wide classes of problems and it increases the accuracy of the solutions.

(Ramadan, et al.,2011) presented an extension and generalization of the polynomial spline functions used in (Ramadan, 2005), to investigate the solution of the fractional ordinary differential equation given by:

$$y^{(\alpha)}(x) = f(x, y(x)) \qquad a \le x \le b$$

$$y(a) = y_0, \qquad \alpha > 1 \qquad (3.1.3)$$

where f is known function, y is the unknown function need to be found for x > a.

The present chapter is a sequel to this work (Ramadan, et al.,2011) and we extend the application of spline functions method of polynomial form to a more general case incorporating nonlinearities as well. And we show that this analysis is also applicable in the case $0 < \alpha \le 1$.

3.2 Existence and uniqueness of the solution

Looking at the questions of existence and uniqueness of the solution, we can present the following results that are very similar to the corresponding classical theorems known in the case of first-order equation. Only the scalar setting will be discussed explicitly; the generalization to vector-valued functions is straightforward.

Theorem 3.1 (Existence) (Ramadan, 2005)

Assume that

D:= $[0,x^*] \times \left[y_0^{(0)} - \propto, y_0^{(0)} + \propto\right]$ *with some* $x^* > 0$ *and some* $\propto > 0$ *and let the function* $f: D \to R$ *be continous*

Furthermore, define x: = min $\{x^*, (\frac{\propto \Gamma(q+1)}{\|f\|_\infty})^{1/q}\}$, Then, there exists a function *y:* [0, x] → *R* solving the equation (3.1.3).

Theorem 3.2 (Uniqueness) (Ramadan, 2005)

Assume that D: $[0,x^*] \times \left[y_0^{(0)} - \propto, y_0^{(0)} + \propto\right]$ *with some* $x^* > 0$ *and some* $\propto > 0$ *and let the function* $f: D \to R$ be bounded on *D* and fulfils a Lipschitz condition with respect to the second variable, i.e.

$$|f(x,y) - f(x,z)| \le L|y - z|$$

with some constant *L* > 0 independent of *x*, *y* and *z* . Then, denoting x as in theorem 3.1, there exists at most one function *y* : [0,χ] → *R* solving the equation (3.1.3).

Proof: The proof as in (Kilbas, et al., 2006).

Suppose that *f* : [*a*,*b*]× *R* → *R* is continuous and satisfies the Lipsechitz condition

$$|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2| \qquad (3.2.1)$$

with $\forall\ (x,y_1),(x,y_2) \in [a,b] \times R$

These conditions assure the existence of unique solution *y* of equation (3.1.3). Let Δ be a uniform partition to the interval [*a*,*b*]defined by the nodes:

$\Delta: a = x_0 < x_1 < x_2 < \cdots < x_k < x_{k+1} \ldots < x_n = b, x_k = x_0 + kh, h = \dfrac{b-a}{k}, k = 0,1 \ldots, n-1$

Define the new form of fractional spline function S(x) of polynomial form approximating the exact solution y by:

$$S_\Delta(x) = S_k(x) = S_{k-1}{}^{(\alpha-1)}(x_k)\frac{(x-x_k)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r} M_k^{(\alpha+i-1)}\frac{(x-x_k)^{i+\alpha}}{\Gamma(\alpha+i+1)} \qquad (3.2.2)$$

Where $M_k^{(\alpha)} = f^{(\alpha)}(x_k, S_{k-1}(x_k))$ with $S_{-1}(x_0) = y_0, for\ x \in [x_k, x_{k+1}]$, Such that S_Δ (x) exists and is unique.

3.3 Error Estimation and Convergence Analysis

To estimate the error, we write the exact solution *y*(*x*) in the Taylor form see (Munkhammar, 2004) and (Ramadan, et al., 2011)

$$y(x) = y_k^{(\alpha-1)} \frac{(x-x_k)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r-1} y_k^{(\alpha+i-2)} \frac{(x-x_k)^{i+\alpha-2}}{\Gamma(\alpha+i+1)} + y_k^{(\alpha-r-1)}(\xi_k) \frac{(x-x_k)^{\alpha-r-1}}{\Gamma(\alpha-r)}, n \in Z^+$$

Where $\xi_k \in [x_k, x_{k+1}] and\ y_k = y(x_k)$

And in this part we use α > 1as in the previous remark. Moreover, we denote to the estimated error of *y*(*x*) at any point *x* ∈ [*a*, *b*] by:

$$e^{(\alpha)}(x) = \left| y^{(\alpha)}(x) - S_\Delta^{(\alpha)}(x) \right| = e^{(\alpha)}{}_k(x) = \left| y^{(\alpha)}{}_k - S_{k-1}^{(\alpha)}(x_k) \right| \qquad (3.3.1)$$

Define the modulus of continuity of $\omega(y^{(\alpha-r-1)}(\xi_k, h))$ as following:

$$\omega\left(y^{(\alpha-r-1)}(\xi_k, h)\right) = Max_{\xi_k \in [a,b]}\{\left|y^{(\alpha-r-1)}(\xi_k + h) - y^{(\alpha-r-1)}(\xi_k)\right|\}$$

(**Hint:** $(x-x_k)^{\alpha+i} < (x-x_k)^{\alpha+i-2}$ since $(x-x_k) < 1\ and\ \Gamma(\alpha+i+1) \geq \Gamma(\alpha+i-1)$.

Implies that $\frac{1}{\Gamma(\alpha+i+1)} \leq \frac{1}{\Gamma(\alpha+i-1)}$.

Then $\sum_{i=-n+1}^{r} M_k^{(\alpha+i-1)} \frac{(x-x_k)^{i+\alpha}}{\Gamma(\alpha+i+1)} \leq \sum_{i=-n+1}^{r} M_k^{(\alpha+i-1)} \frac{(x-x_k)^{i+\alpha-2}}{\Gamma(\alpha+i-1)}$

So that we can write

$$S_\Delta(x) = S_k(x) \cong S_{k-1}{}^{(\alpha-1)}(x_k) \frac{(x-x_k)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r} M_k^{(\alpha+i-1)} \frac{(x-x_k)^{i+\alpha-2}}{(\alpha+i-1)!} \qquad (3.3.2)$$

Lemma 1

Let e(x) be defined as in (3.3.1) then there exist constant d independent of h such that the following inequality:

$$e(x) \leq e_k^{(\alpha-1)} \frac{h^{\alpha-1}}{\Gamma(\alpha)} + d e_k h^{\alpha+i-2} + \omega(h) \frac{h^{\alpha-r-1}}{\Gamma(\alpha-r)} \quad holds\ for\ all\ x \in [a,b]$$

Proof:

Using the a Lipschitz condition, Taylor expansion, definition of error estimation and (3.3.2) we get:

$$e(x) = |y(x) - S_\Delta(x)| \cong \left| \left(y_k^{(\alpha-1)} - S_{k-1}^{(\alpha-1)}\right) \frac{(x-x_k)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r-1} y_k^{(\alpha+i-2)} \frac{(x-x_k)^{i+\alpha-2}}{\Gamma(\alpha+i-1)} - \sum_{i=-n+1}^{r-1} M_k^{(\alpha+i-1)} \frac{(x-x_k)^{i+\alpha-2}}{(\alpha+i-1)!} + y_k^{(\alpha-r-1)}(\xi_k) \frac{(x-x_k)^{\alpha-r-1}}{\Gamma(\alpha-r)} - M_k^{(\alpha-r)}(\xi_k) \frac{(x-x_k)^{\alpha-r-1}}{\Gamma(\alpha-r)} \right|$$

$$\leq e_k^{(\alpha-1)} \frac{|(x-x_k)^{\alpha-1}|}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r-1} \left| y_k^{(\alpha+i-2)} - M_k^{(\alpha+i-1)} \right| \frac{|(x-x_k)^{i+\alpha-2}|}{\Gamma(\alpha+r-1)} + \left| y_k^{(\alpha-r-1)}(\xi_k) - M_k^{(\alpha-r)} \right| \frac{|(x-x_k)^{\alpha-r-1}|}{\Gamma(\alpha-r)}, \qquad (3.3.3)$$

Where

$$\left| y_k^{(\alpha+i-2)} - M_k^{(\alpha+i-1)} \right| = |f(x_k, y_k) - f(x_k, S_{k-1}(x_k))| \leq L\{|y_k - S_{k-1}(x_k)|\} \leq Le_k$$

Similarly

$$\left| y_k^{(\alpha-r-1)}(\xi_k) - M_k^{(\alpha-r)} \right| \leq \left| y^{(\alpha-r-1)}(\xi_k) - y_k^{(\alpha-r-1)} \right| + \left| y_k^{(\alpha-r-1)}(\xi_k) - M_k^{(\alpha-r)} \right| \leq \omega(h) + Le_k$$

where constant $L > 0$ is the Lipsechitz constant independent of h and $\omega(h)$ is the modulus of Continuity$of\ \omega\left(y^{(\alpha-r-1)}(\xi_k, h)\right)$. The inequality (3.3.3) is then reduced to

$$e(x) \leq e_k^{(\alpha-1)} \frac{\left|(x-x_k)^{\alpha-1}\right|}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r-1} Le_k \frac{\left|(x-x_k)^{\alpha+i-2}\right|}{\Gamma(\alpha+r-1)} + (\omega(h) + Le_k) \frac{\left|(x-x_k)^{\alpha-r-1}\right|}{\Gamma(\alpha-r)},$$

$$\leq e_k^{(\alpha-1)} \frac{h^{\alpha-1}}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r-1} \frac{Le_k h^{\alpha+r-2}}{\Gamma(\alpha+i-1)} + (\omega(h) + Le_k) \frac{h^{\alpha-r-1}}{\Gamma(\alpha-r)}$$

$$\leq e_k^{(\alpha-1)} \frac{h^{\alpha-1}}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r} \frac{Le_k h^{\alpha+i-2}}{\Gamma(\alpha+i-1)} + \omega(h) \frac{h^{\alpha-r-1}}{\Gamma(\alpha-r)}$$

$$= e_k^{(\alpha-1)} \frac{h^{\alpha-1}}{\Gamma(\alpha)} + d\ e_k h^{\alpha+i-2} + \omega(h) \frac{h^{\alpha-r-1}}{\Gamma(\alpha-r)}$$

Setting $x = x_{k+1}$, we get:

$$e_{k+1} \leq e_k^{(\alpha-1)} \frac{h^{\alpha-1}}{\Gamma(\alpha)} + de_k h^{\alpha+i-2} + \omega(h) \frac{h^{\alpha-r-1}}{\Gamma(\alpha-r)} \quad where\ d = \sum_{i=-n+1}^{r} \frac{L}{\Gamma(\alpha+i-1)}$$

Is constant independent of h.

3.4 Stability Analysis of the Method

For analyzing the stability properties of the given method, we make a small change of the starting values and study the changes in the numerical solution produced by the method.

Now, we define the spline approximating function $\tilde{S}_{\Delta}(x)$ as:

$$\tilde{S}_{\Delta}(x) = \tilde{S}_k(x) = \tilde{S}_{k-1}{}^{(\alpha-1)}(x_k)\frac{(x-x_k)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r} \tilde{M}_k^{(\alpha+i-1)} \frac{(x-x_k)^{i+\alpha}}{\Gamma(\alpha+i+1)} \qquad (3.4.1)$$

Where $\tilde{M}_k^{(\alpha)} = f^{(\alpha)}\left(x_k, \tilde{S}_{k-1}(x_k)\right)$

with $\tilde{S}_{-1}(x_0) = y^*{}_0, for\ x \in [x_k, x_{k+1}]$, *and r=0, 1,..,n*

and use the notation

$$e^{(\alpha)^*}(x) = \left|S_{\Delta}{}^{(\alpha)}(x) - \tilde{S}_{\Delta}^{(\alpha)}(x)\right|, and\ e^{(\alpha)^*}{}_k = \left|S^{(\alpha)}{}_k - \tilde{S}_{k-1}^{(\alpha)}(x_k)\right| \qquad (3.4.2)$$

Lemma 2 (Ramadan, 2005)

Let $e^*(x)$ be defined as In (3.4.1), then the inequality

$$e^*(x) \le e_k^{(\alpha-1)^*} \frac{h^{\alpha-1}}{\Gamma(\alpha)} + \tilde{d} e_k{}^* h^{\alpha+i}\ where\ e_0^* = |y_0 - y_0^*|$$

Holds where $\tilde{d}$ is a constant independent of h.

Proof

Using Lipsechitz condition and (3.2.2), (3.4.1) and (3.4.2) we get:

$$e^*(x) = \left|S_{\Delta}^{(\alpha)}(x) - \tilde{S}_{\Delta}{}^{(\alpha)}(x)\right|$$

$$= \left|(s_k^{(\alpha-1)} - \tilde{s}_{k-1}^{(\alpha-1)})\frac{(x-x_k)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r} M_k^{(\alpha+i-1)}\frac{(x-x_k)^{\alpha+i}}{\Gamma(\alpha+i+1)} - \sum_{i=-n+1}^{r} \tilde{M}_k^{(\alpha+i-1)}\frac{(x-x_k)^{\alpha+i}}{\Gamma(\alpha+i+1)}\right|$$

$$\le e_k^{(\alpha-1)^*} \frac{\left|(x-x_k)^{\alpha-1}\right|}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r} \left|M_k^{(\alpha+i-1)} - \tilde{M}_k^{(\alpha+i-1)}\right| \frac{\left|(x-x_k)^{\alpha+i}\right|}{\Gamma(\alpha+i+1)}$$

$$\le e_k^{(\alpha-1)^*} \frac{h^{\alpha-1}}{\Gamma(\alpha)} + \sum_{i=-n+1}^{r} \left|M_k^{(\alpha+i-1)} - \tilde{M}_k^{(\alpha+i-1)}\right| \frac{h^{\alpha+r}}{\Gamma(\alpha+r+1)}$$

(3.4.3)

But $\left|M^{(\propto+i-1)}{}_{k} - \widetilde{M}_{k}^{(\propto+i-1)}\right| \leq \tilde{d}\, e_{k}^{*}$ (3.4.4)

Thus from (3.4.3) and (3.4.4) we obtain:

$$e^{*}(x) \leq e_{k}^{(\propto-1)^{*}} \frac{h^{\propto-1}}{\Gamma(\propto)} + \tilde{d} e_{k}{}^{*} h^{\propto+i}$$

$where\ \ \tilde{d} = \sum_{i=-n+1}^{r} \frac{L}{\Gamma(\alpha+i-1)}$ is constant independent of h.

Setting $x = x_{k+1}, we\ get:$

$$e^{*}{}_{k+1}(x) \leq e_{k}^{(\propto-1)^{*}} \frac{h^{\propto-1}}{\Gamma(\propto)} + \tilde{d} e_{k}{}^{*} h^{\propto+i}$$

Which prove the lemma.

3.5 Numerical Examples and Conclusion

To demonstrate the effectiveness of this scheme, we consider two kinds of systems. These examples are considered because closed form solutions are available for them in the case of integer-order derivatives, or they have also been solved using other numerical schemes. This allows one to compare the results obtained using this scheme with the analytical solution or the solutions obtained using other schemes**.**

Example 1

Consider the fractional differential equation

$$\frac{d^{\alpha}y}{dx^{\alpha}} = -y^{2}(x) + 1,$$

$$The\ exact\ solution\ is\ y(x) = \frac{e^{2x}-1}{e^{2x}+1}\ \ ,and\ initial\ condition\ y(0) = 0$$

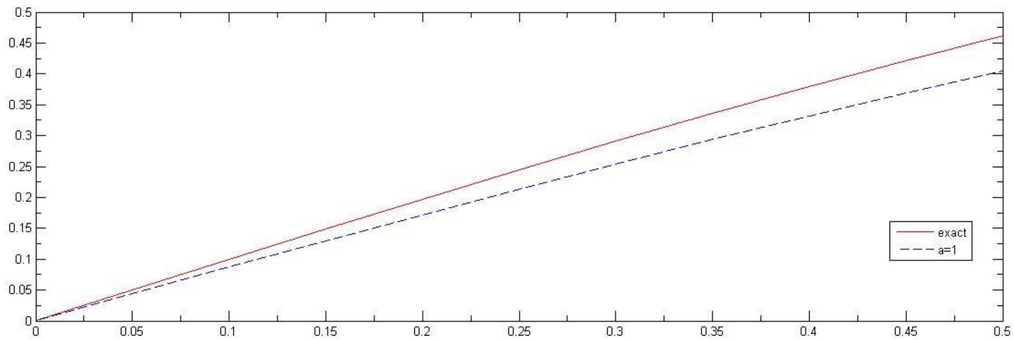


Figure 3.1: The exact solution and Approximation solution at α=1 for example 1

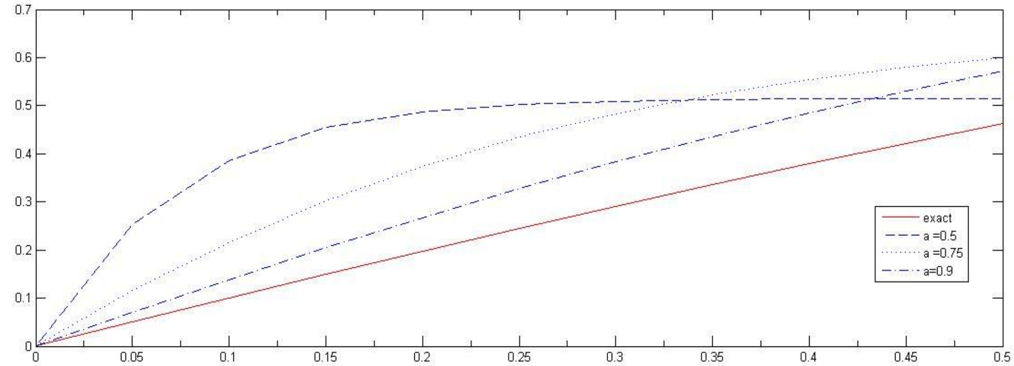


Figure 3.2: The exact solution and Approximation solution at α=0.5, 0.75 and 0.9 for example 1

Example 2

Consider the fractional differential equation

$$\frac{d^{\alpha}y}{dx^{\alpha}} = 2y(x) - y^2(x) + 1,$$

The exact solution is

$$y(x) = 1 + \sqrt{2}\tanh\left(\sqrt{2}x + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right), \text{and initial condition } y(0) = 0$$

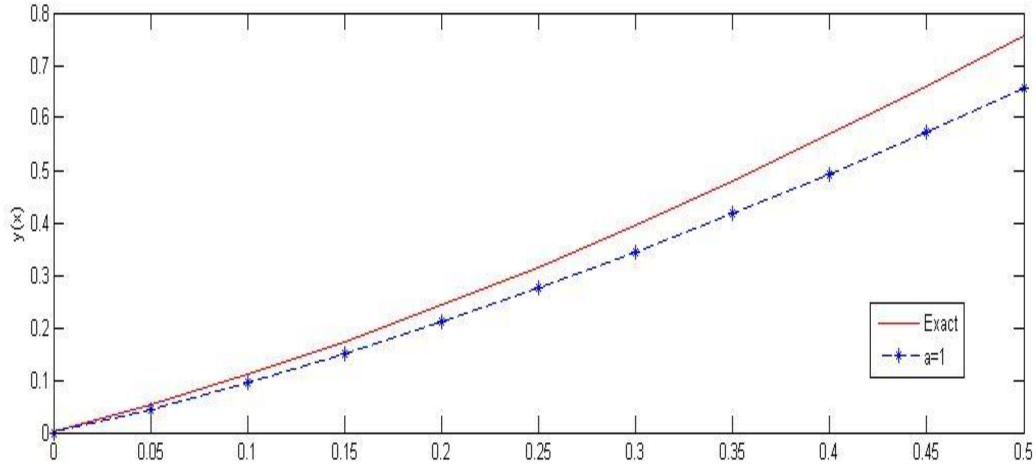


Figure 3.3: The exact solution and Approximation solution at α=1 for example 2

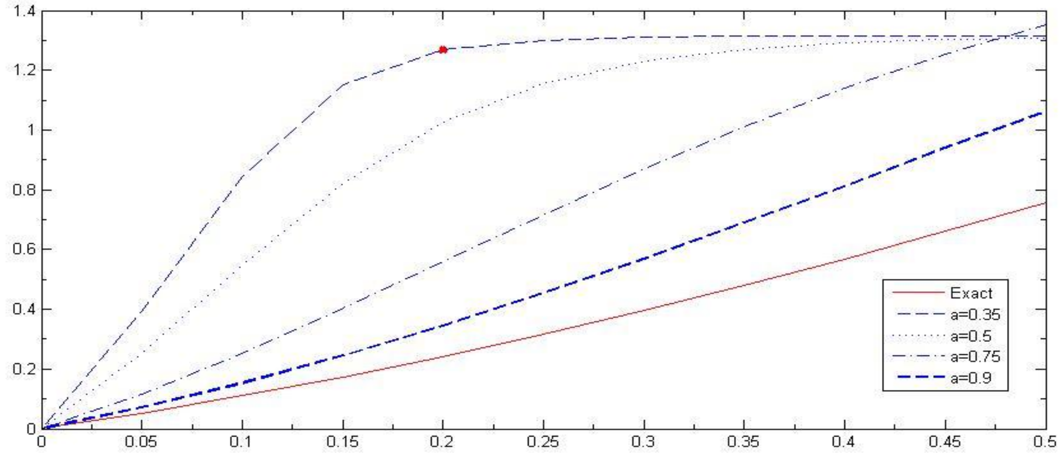


Figure 3.4: The exact solution and Approximation solution at α=0.35, 0.5, 0.75, and 0.9 for example 2

Conclusion

In this thesis, the spline functions was utilized successfully to find accurate approximate solutions of fractional differential equations. We investigated the possibility of using the same spline functions given in (Ramadan, 2005) to be in fractional form with some additional assumptions and definitions for approximating the solution of fractional ordinary differential equation. Numerical examples are given to illustrate the applicability, accuracy and stability of the proposed method. We applied the proposed technique to two nonlinear examples to test the accuracy and stability. The obtained numerical results reveal that the method is stable and gives high accuracy.

The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. Many of the results obtained in this thesis can be extended to significantly more general classes of linear and non-linear differential equations of fractional order.

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تطبيق الاقترانات المفتاحية للمعادلات التفاضلية الكسرية

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الملخص

تتناول الرسالة الحل التقريبي للمعادلات التفاضلية الكسرية الخطية واللاخطية باستخدام الاقترانات المفتاحية المتعددة الحدود، ولقد قدم وصفا للطريقة المقترحة وتم التحقق من تحليل الخطأ وأسلوب الاستقرار من الناحية النظرية، إن الامثلة العددية توضح مدى الانطباق والاستقرار في الطريقة المقترحة حيث عرفت المشتقات الكسرية بمعنى كابوتو، وتظهر النتائج التي تم الحصول عليها على توافق جيد مع الحلول التحليلية والنتائج العددية التي عرضت من قبل.

توضح النتائج التي تم الوصول إليها أن طريقة الاقترانات المفتاحية قوية جدا وفعالة في ايجاد حلول عددية للمعادلات التفاضلية الكسرية مما تزيد من دقة الحلول، ومن المتوقع أن الطريقة المقترحة يمكن استخدامها في حل مشاكل خطية مماثلة في حساب التفاضل والتكامل الكسري.